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Radar error sensitivity coeeficients of lunar landing navigation

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by

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ABSTRACT

This paper describes the investigation of radar measurement errors on the performance of a lunar landing guidance technique. The radar measurements are used to estimate the position and velocity of the landing vehicle relative to a radar beacon located at the desired landing site.

A linearized analysis is performed, resulting in a set of error sensitivity coefficients. These coefficients relate the statistics of position and velocity errors at the termination of the descent maneuver to the statistics of the radar sensor errors which caused them. A sample computation is performed to indicate the application of these sensitivity coefficients.

Acknowledgements

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I wish to acknowledge the counsel of my associates at Westinghouse principal among whom have been Mr. George S. Axelby, an advisory engineer at Westinghouse; Mr. John W. Knight, my immediate supervisor; and Mr. James L. Farrell, another member of the contract team. In addition I would like to thank Professor John J. Karakash of Lehigh for his continuing interest, encouragement and valuable suggestions. Finally, a word of thanks to Miss Grace Ivers who is responsible for the major portion of final draft typing effort.

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1.0 INTRODUCTION

When one contemplates the problems involved in soft-landing a spacecraft on the lunar surface, it soon becomes apparent that direct observation of spacecraft position and velocity relative to the lunar surface (and more specifically, relative to the desired landing site) will be required during the terminal phase of such a venture. One technique available for obtaining the required information is radar.

The realization that direct observation of vehicle position and velocity is required and the generation of an information sensing system concept to provide the necessary information is not the complete solution to this problem. One must realize that the sensing system will not deliver perfect information to the guidance and control system. Noise in the sensing system will cause error in the estimated vehicle state*, which in turn will cause erroneous thrust vector commands in the engine control system. Thrust vector errors acting over a period of time produce position and velocity errors. Thus sensor errors will propagate through the guidance and control system and produce guidance errors.

When designing a guidance and control system for landing, it is important that the position and velocity errors existing at the end of the landing maneuver caused by sensor errors injected into the guidance system during the performance of the maneuver not be large enough to endanger successful landing.

The object of this paper is to develop sensitivity coefficients relating sensor errors to the resulting position and velocity errors at the end of the

* Vehicle state refers to the position and velocity of the spacecraft. In general there are six quantities which completely define vehicle state at any given time, three position components and three velocity components.

landing maneuver given a particular information sensing system; guidance technique; and nominal trajectory. A detailed description of these three items constitutes the problem definition section of this report. For introductory purposes the following statements are useful:

- Information sensing system: A tracking radar locked on a cooperative beacon located at the desired landing site. The measured quantities are line-of-sight range (R) and angle (ϕ) from the spacecraft to the beacon. The time derivatives of these two quantities are also measured. The four variables R, $\dot{R}$, ϕ, and $\dot{\phi}$ are referred to as observables.
- Nominal trajectory: The nominal trajectory is a constant thrust gravity turn which terminates with the spacecraft at zero velocity 500 meters directly above the desired landing site.
- Guidance technique: A linearized, predictive guidance technique is employed. The guidance system compares estimated vehicle state variables with stored nominal values and then predicts what terminal errors will result if no corrective action is taken. (Time after thrust initiation is used as a basis for comparison of estimated and nominal vehicle state variables.) A thrust vector correction which will reduce the estimated terminal position errors to zero is then calculated and transmitted to the engine control system.

Following the definition section is the analytical section. Here the equations yielding error sensitivity coefficients are derived in a form suitable for evaluation with the aid of a digital computer. Finally, the computer results are tabulated and their use illustrated by example.

2.0 PROBLEM DEFINITION

The following brief sections serve to establish an environment within which the effects of radar sensor errors on landing vehicle performance can be assessed. No attempt at optimization has been made in these areas; however, the particular techniques discussed have been chosen from those that are available for specific reasons. These reasons are included in the discussion where pertinent.

2.1 Analytical Assumptions

There are four assumptions made with regard to vehicle dynamics that facilitate the error analysis.

- The motion of the vehicle can be adequately described by the dynamics of a restricted two body system. This assumption, which is validated by the fact that space vehicle mass is negligible compared to the lunar mass, is commonly used to describe the motion of a spacecraft operating in the vicinity of a massive body.
- No dynamic perturbations such as those caused by lunar oblateness or gravitational anomalies are to be considered. The presence or absence of these effects in the analytical model will not alter the effects of sensor errors and therefore need not be considered.
- The moon is considered to be stationary during the performance of the descent maneuver. The actual rotational motion of the moon during the time occupied is not negligible. However, the coordinate system used during the landing is referenced to a point fixed to the lunar surface. Measurements of quantities in this reference system will not be influenced by the rotation so it might as well be set to zero.
- The landing trajectory is assumed to be two-dimensional. That is, the plane formed by the vehicle position vector, in lunar central

coordinates; and the vehicle velocity vector is assumed to contain the desired landing site. In practice, deviations of the target landing site ~~from the~~ plane defined above are expected to be small. These are the basic analytical assumptions.

It might also be appropriate to point out that this analysis does not assume a flat moon or a uniform gravitational field. Trajectory determination was done using a spherical moon and a central gravitational field.

2.2 State Variables

The first quantities requiring definition are those which describe the space vehicle position and velocity at any time during the landing maneuver. Ordinarily, 6 quantities are needed to specify the vehicle state completely; 3 position vector components and 3 velocity vector components. However, restriction of the landing geometry to two dimensions means that the instantaneous state can be described completely by a total of 4 state variables. The state variables that have been selected are defined below and illustrated in figure 1.

- h : The altitude of the craft above a reference sphere. The radius of the reference sphere is equal to the lunar radius at the target landing site; meters.
- θ : The angular displacement of the spacecraft from the target landing site in lunar central coordinates; radians.
- γ : The velocity vector angle relative to the landing vehicle local horizontal plane, measured in the plane of the landing trajectory; radians.
- V : The magnitude of the velocity vector, meters/second.

Quantities h and θ are seen to define spacecraft position while γ and V specify the velocity vector.

2.3 Guidance Technique

To evaluate the effects of error in the measurement of observable quantities, one must know how the observed information is to be used. With

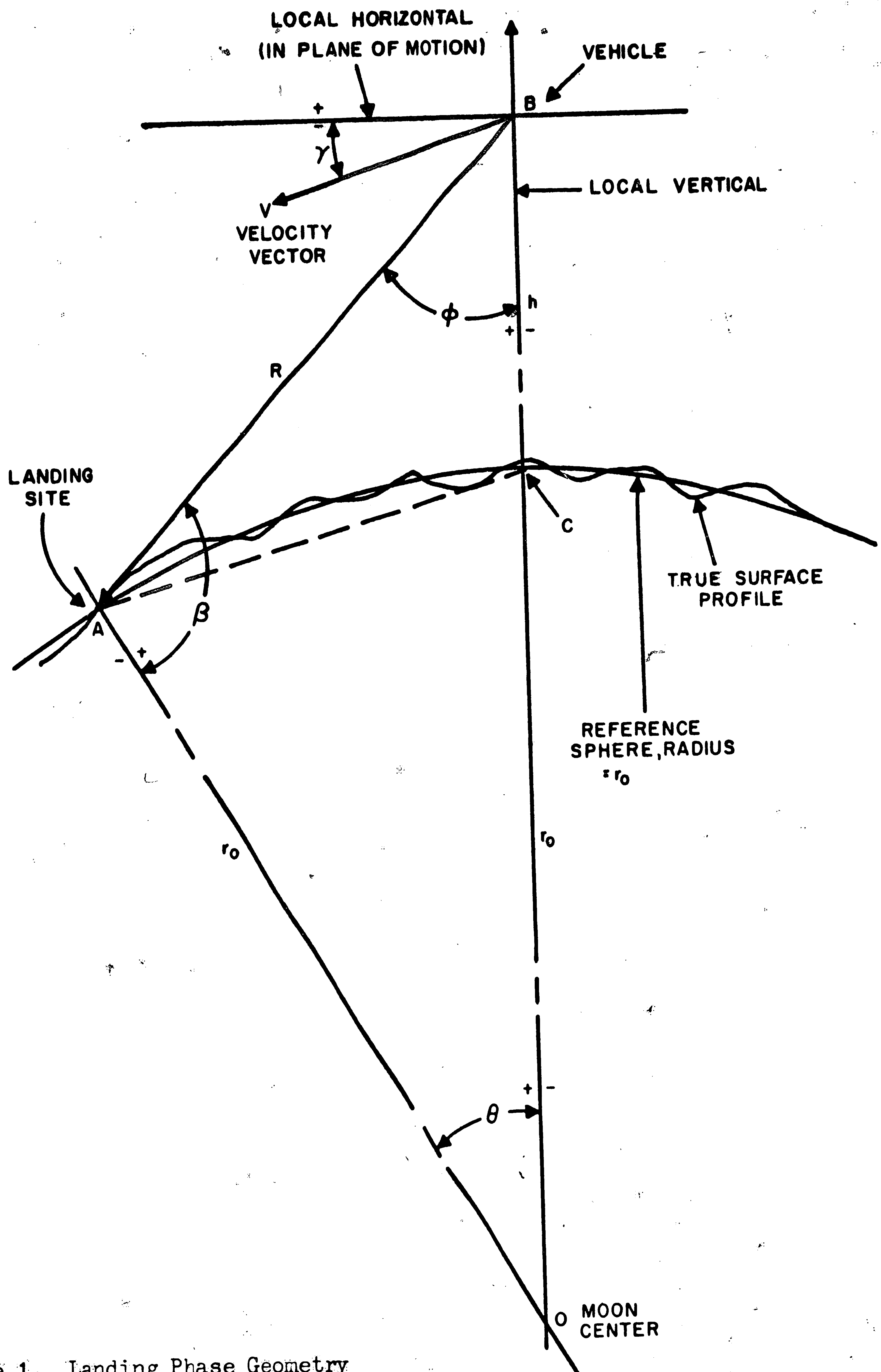

Figure 1. Landing Phase Geometry

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regard to the present discussion, this implies that knowledge of the guidance system technique is required. The particular guidance scheme finally selected was chosen on the basis of its versatility. It is capable of correcting relatively large deviations from nominal initial conditions at reasonable fuel expense*. In addition, the guidance technique can be used in either manned or unmanned vehicles. The discussion of the guidance technique included here is qualitative in nature. For quantitative derivation of the guidance equations see Appendix A.

The system can be classified as a linear predictive guidance system which functions in the following manner.

1) Radar information is fed to a computer which turns out an estimate of the vehicle state at time <u>t</u>. (Time is referenced to the time of landing engine ignition.)

2) Estimated state variables are compared with reference values that would exist at time <u>t</u> if the vehicle were on the reference trajectory.

3) Differences between estimated and reference state variables are inputs to linearized equations which estimate terminal deviations that will exist unless some deviation from the reference thrusting program is implemented.

4) The magnitude of the required thrust correction is computed and fed to the engine control system.

5) The process is continuous throughout the landing maneuver.

---

*Reference 1 is a report dealing with the capabilities of the guidance system which may be referred to if more information is desired.

In practice, most of the computation indicated in steps 3 and 4 is based on linearization about the reference trajectory and, hence, can be done prior to flight. The result is a set of guidance terms denoted S_1 through S_8 which multiply the estimated state variable errors directly to yield thrust corrections. Figure 2 is a block diagram of the guidance system. Notice that there are two thrust control quantities: T, the thrust vector magnitude; and α, the thrust vector direction (see figure 3). Only one angle is required because of the restriction to a two-dimensional analysis.

Recall that linearized equations are employed to predict terminal deviations from the reference trajectory, and therefore, to compute the thrust vector corrections. Linearization error, which is bound to exist, will result in thrust correction errors; which in turn will be reflected in terminal deviations from the reference trajectory. Terminal deviations caused by linearization error exist even if sensor information is perfect. The terminal error caused by linearization is not random, but rather, is completely specified once the exact point of landing engine ignition is specified. To account for this linearization error it is convenient to introduce the concept of a pseudo-reference trajectory which is defined as the trajectory the vehicle would fly if sensor information were perfect. If sensor information is noisy, the actual trajectory will wander from the pseudo-reference trajectory in a random manner, the distribution of actual endpoints corresponding to a given set of initial conditions. The object of this work is to derive estimates of the statistical parameters which describe the spread of the distribution of actual endpoints about the pseudo-reference endpoint.

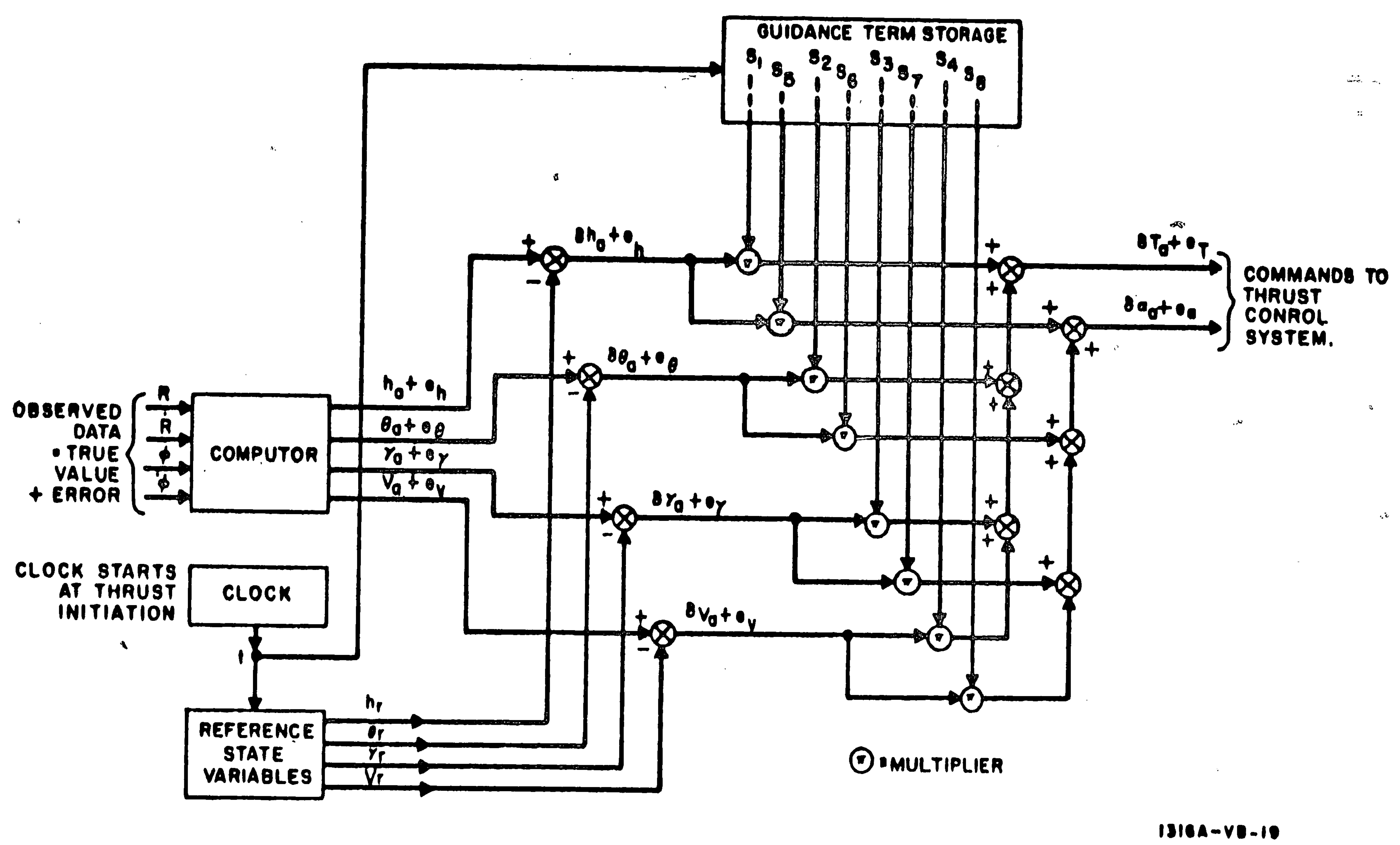

Figure 2. Block Diagram of Guidance System Analytical Model

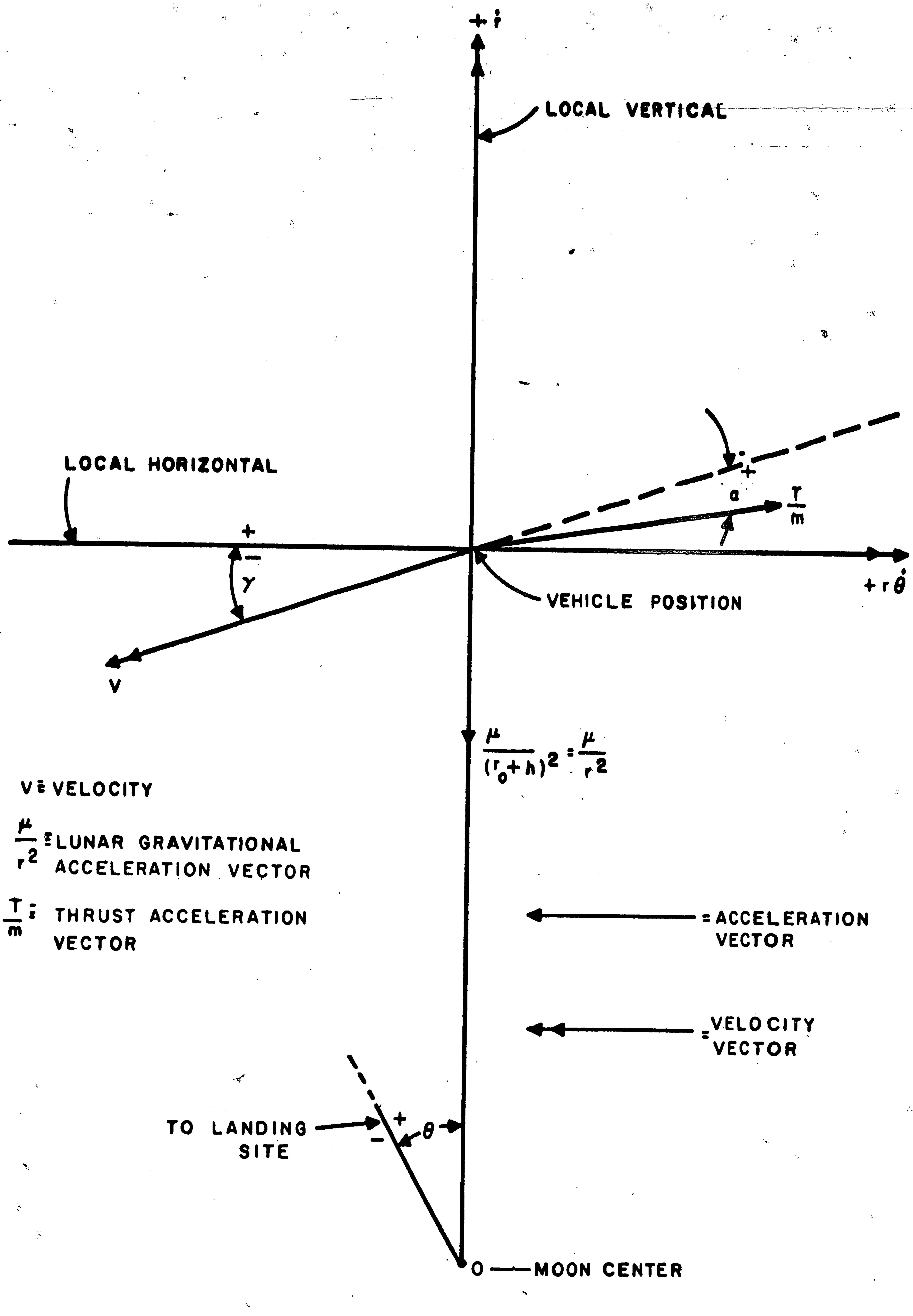

Figure 3. Velocity-Acceleration Diagram of Landing Maneuver

2.4 Reference Trajectory

It has been pointed out previously that the guidance law selected requires the determination of a reference trajectory. The values of h, θ, γ, V, and the eight guidance terms stored in the guidance computer are calculated beforehand, based on the reference path. It was decided to use a constant thrust, gravity turn trajectory commencing at periselenum of an elliptical parking orbit, and terminating with the vehicle in a hover condition directly above the desired landing site. There are several advantages to this type of trajectory.

1) Continuous thrust is preferable to impulsive thrust because the use of a two-or three-impulse landing scheme would imply high acceleration levels and large engines. The use of large engines will incur a weight penalty over the smaller engines required for a continuous thrust trajectory. Furthermore, continuous thrust does not demand the shutdown and restart capabilities of a multiple impulse landing;

2) A constant, or approximately constant thrust level provides the minimum engine size, and minimizes throttling requirements during the descent trajectory;

3) A gravity turn trajectory is near optimum from the standpoint of fuel consumption because the thrust is always in the direction to reduce velocity most effectively;

4) The gravity turn trajectory automatically rotates the vehicle to the proper hovering attitude during the descent so that no last second, high rate rotations are required.

Since knowledge of reference trajectory parameters is required for the error analysis, a computer program was developed to provide the necessary information. Given initial values of vehicle velocity (V_o), attitude angle (γ_o), and altitude (h_o), this program computes the initial value of angular displacement from the landing site (θ_o) and the thrust level (T_o) required to produce a hover state directly over the target landing site at an altitude of 500 meters.

The method of this program is to assume a complete set of initial conditions, integrate the equations of motion forward in time until V = 0, and compare the computed values of h_f and γ_f corresponding to V = 0 with the desired reference values of h_f and γ_f which are 500 meters and -90^o respectively. This computed deviation from reference hover state provides information helpful in determining changes to be made in the assumed initial conditions before making the next run through the program.

The general equations of motion used for the reference trajectory determination program are:

$$
\begin{aligned}
&(a)\quad \dot{h} = V\sin\gamma \\
&(b)\quad \dot{\theta} = -\frac{V}{r_o + h}\cos\gamma \\
&(c)\quad \dot{V} = \frac{T\cos\alpha}{m_o - \int_o^t kT\,dt} - \frac{\mu\sin\gamma}{(r_o+h)^2} \\
&(d)\quad \dot{\gamma} = \frac{V\cos\gamma}{(r_o+h)} - \frac{T\sin\alpha}{V(m_o - \int_o^t kT\,dt)} - \frac{\mu\cos\gamma}{V(r_o+h)^2}
\end{aligned}
\qquad (2\text{-}1)
$$

where h, θ, γ, and V have been defined previously, and:

r_o = radius of reference lunar sphere.

T = thrust vector magnitude.

α = angle between vehicle velocity vector and thrust vector.

m_o = vehicle mass at time of landing trajectory initiation.

k = constant of proportionality relating instantaneous thrust vector magnitude to instantaneous mass rate of flow.

μ = gravitational constant of the moon.

For the constant thrust, gravity turn reference trajectory,

$T = T_o$; and $\alpha = 0$

Substituting these values into the equations of motion produces the following set of expressions.

$$
\begin{aligned}
&(a) \quad \dot{h} = V \sin \gamma \\
&(b) \quad \dot{\theta} = -\frac{V \cos \gamma}{(r_o + h)} \\
&(c) \quad \dot{V} = -\frac{T_o}{m_o - kT_o t} - \frac{\mu \sin \gamma}{(r_o + h)^2} \\
&(d) \quad \dot{\gamma} = \frac{V \cos \gamma}{r_o + h} - \frac{\mu \cos \gamma}{V(r_o + h)^2}
\end{aligned}
\qquad (2\text{-}2)
$$

These are the equations mechanized in the trajectory determination program.

There are three classes of outputs directly available from the program. First are the initial values of T and θ required to complete the specification of the reference trajectory. Secondly, values of the state variables can be printed out at any desired time along the reference flight path. Finally, two matrices of partial derivatives, relating deviations in state variable derivatives to deviations in state variables; and to deviations in the thrust control quantities can be printed out at specified time intervals along the trajectory.

If the operator δ is used to represent a deviation in the value of the parameter on which it operates, the matrices of partial derivatives mentioned above are contained in the following matrix equation.

$$\begin{bmatrix} \delta\dot{h} \\ \delta\dot{\theta} \\ \delta\dot{\gamma} \\ \delta\dot{V} \end{bmatrix} \doteq \begin{bmatrix} \frac{\partial\dot{h}}{\partial h} & \frac{\partial\dot{h}}{\partial\theta} & \frac{\partial\dot{h}}{\partial\gamma} & \frac{\partial\dot{h}}{\partial V} \\ \frac{\partial\dot{\theta}}{\partial h} & \frac{\partial\dot{\theta}}{\partial\theta} & \frac{\partial\dot{\theta}}{\partial\gamma} & \frac{\partial\dot{\theta}}{\partial V} \\ \frac{\partial\dot{\gamma}}{\partial h} & \frac{\partial\dot{\gamma}}{\partial\theta} & \frac{\partial\dot{\gamma}}{\partial\gamma} & \frac{\partial\dot{\gamma}}{\partial V} \\ \frac{\partial\dot{V}}{\partial h} & \frac{\partial\dot{V}}{\partial\theta} & \frac{\partial\dot{V}}{\partial\gamma} & \frac{\partial\dot{V}}{\partial V} \end{bmatrix} \times \begin{bmatrix} \delta h \\ \delta\theta \\ \delta\gamma \\ \delta V \end{bmatrix} + \begin{bmatrix} \frac{\partial\dot{h}}{\partial T} & \frac{\partial\dot{h}}{\partial\alpha} \\ \frac{\partial\dot{\theta}}{\partial T} & \frac{\partial\dot{\theta}}{\partial\alpha} \\ \frac{\partial\dot{\gamma}}{\partial T} & \frac{\partial\dot{\gamma}}{\partial\alpha} \\ \frac{\partial\dot{V}}{\partial T} & \frac{\partial\dot{V}}{\partial\alpha} \end{bmatrix} \times \begin{bmatrix} \delta T \\ \delta\alpha \end{bmatrix} \tag{2-3}$$

If the deviations δh, $\delta\theta$, $\delta\gamma$, and δV are allowed to equal $\delta\xi_1$, $\delta\xi_2$, $\delta\xi_3$, and $\delta\xi_4$ respectively, the matrix expression can be written more compactly as

$$[\delta\dot{\xi}_i] \doteq [A_{ij}][\delta\xi_j] + [B_{ij}]\begin{bmatrix} \delta T \\ \delta\alpha \end{bmatrix}$$

where

$$A_{ij} = \frac{\partial\dot{\xi}_i}{\partial\xi_j}$$

and

$$B_{ij} = \left.\frac{\partial\dot{\xi}_i}{\partial T}\right|_{j=1} ; \quad \left.\frac{\partial\dot{\xi}_i}{\partial\alpha}\right|_{j=2}$$

This is a linearized relationship which is valid only in a small region near the trajectory on which the partial derivatives are evaluated.

To determine the reference trajectory that was used during the remainder of this study, the following fixed initial and final conditions were set into the computer program.

h_o = 20.0 kilometers

γ_o = 0.0 radians

V_o = 1752 meters/second

$t_o = 0.0$ seconds

$h_f = 500$ meters

$V_f = 0.0$ meters/second

$\gamma_f = -\pi/2$ radians

$\theta_f = 0.0$ radians

The specification $\gamma_o = 0$ implies that the landing phase begins at periselenum of a lunar parking orbit.

The following values were used for the various constants required by the program:

$m_o = 11{,}340$ kilograms

$\mu = 0.489820 \times 10^{13}$ meters^3/second^2

$r_o = 1739$ kilometers (mean lunar radius)

$k = 2.54929 \times 10^{-4}$ seconds/meter

(based on specific impulse = 400 seconds.)

The remaining initial and final values of the state variables as determined on the computer are:

$T_o = 48{,}061$ Newtons

$\theta_o = 0.18028$ radians

$t_f = 346.52$ seconds

Some of the pertinent characteristics of the reference trajectory are plotted in figures 4 through 7.

2.5 Conversion of Radar Data to State Variable Information

Before the raw radar input information can be useful to the guidance system, it must be converted to the language of the guidance system. In short, expressions must be found relating the measured quantities to the state variables.

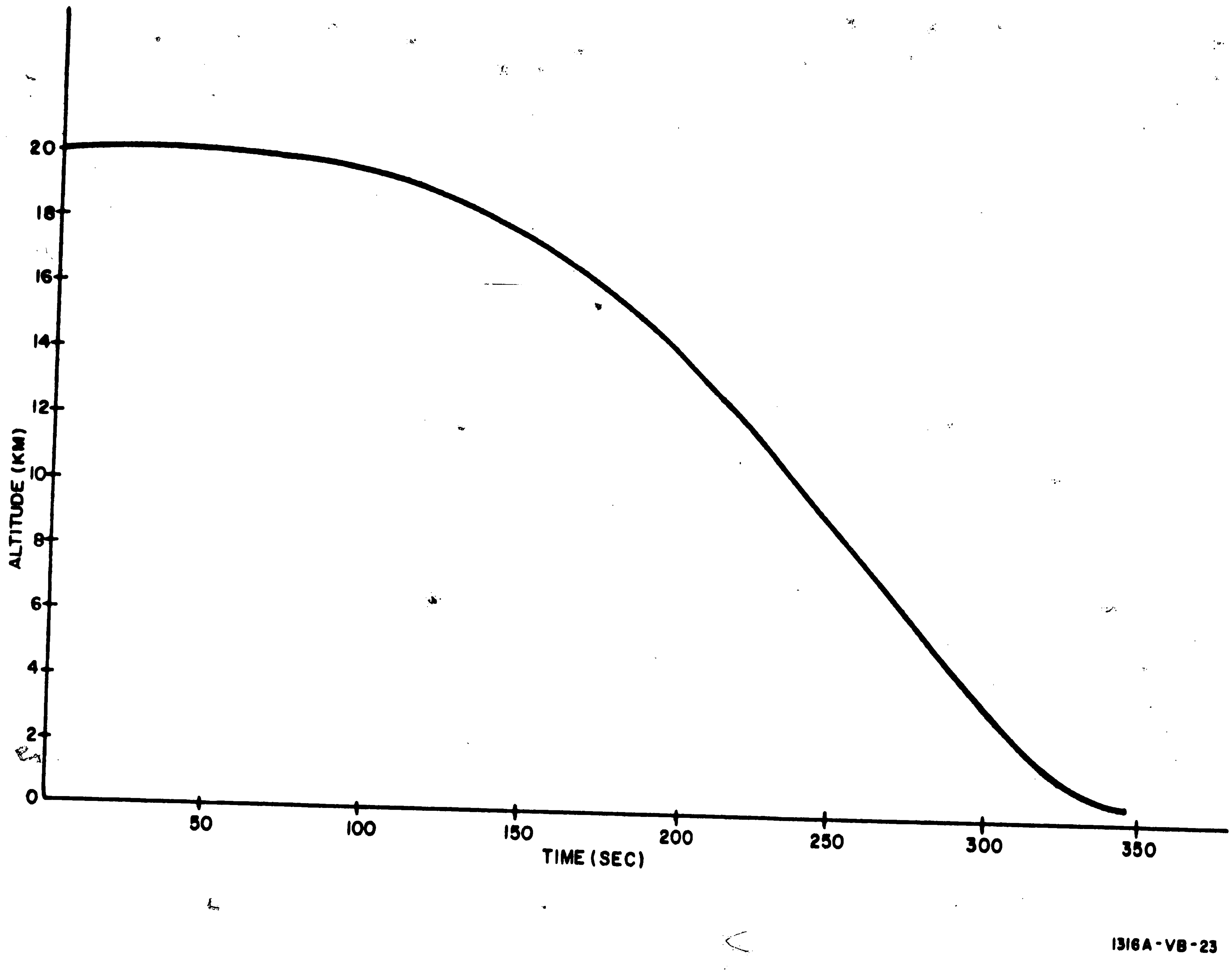

Figure 4. Reference Trajectory: Altitude vs. Time

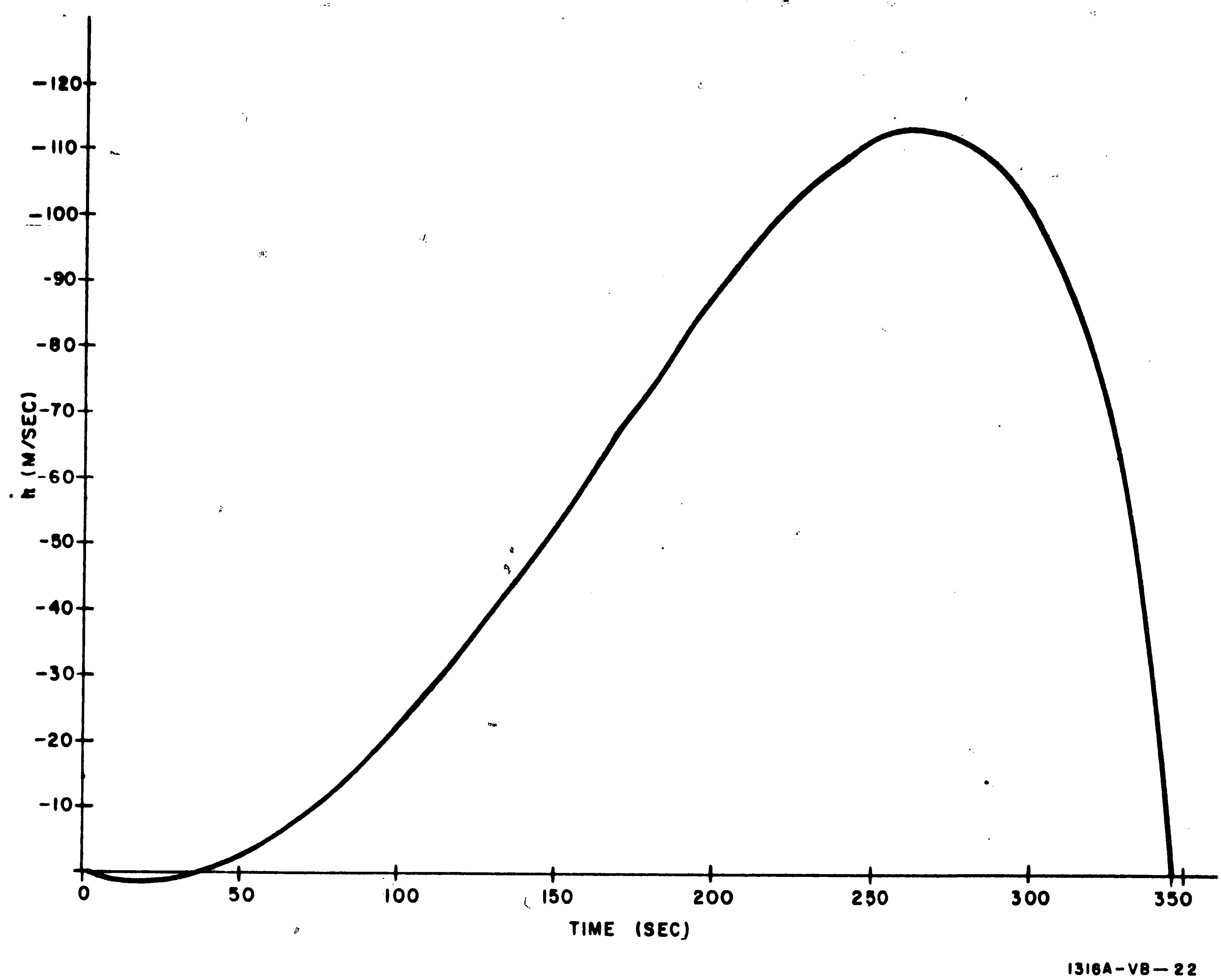

Figure 5. Reference Trajectory: Altitude Rate (ḣ) vs Time

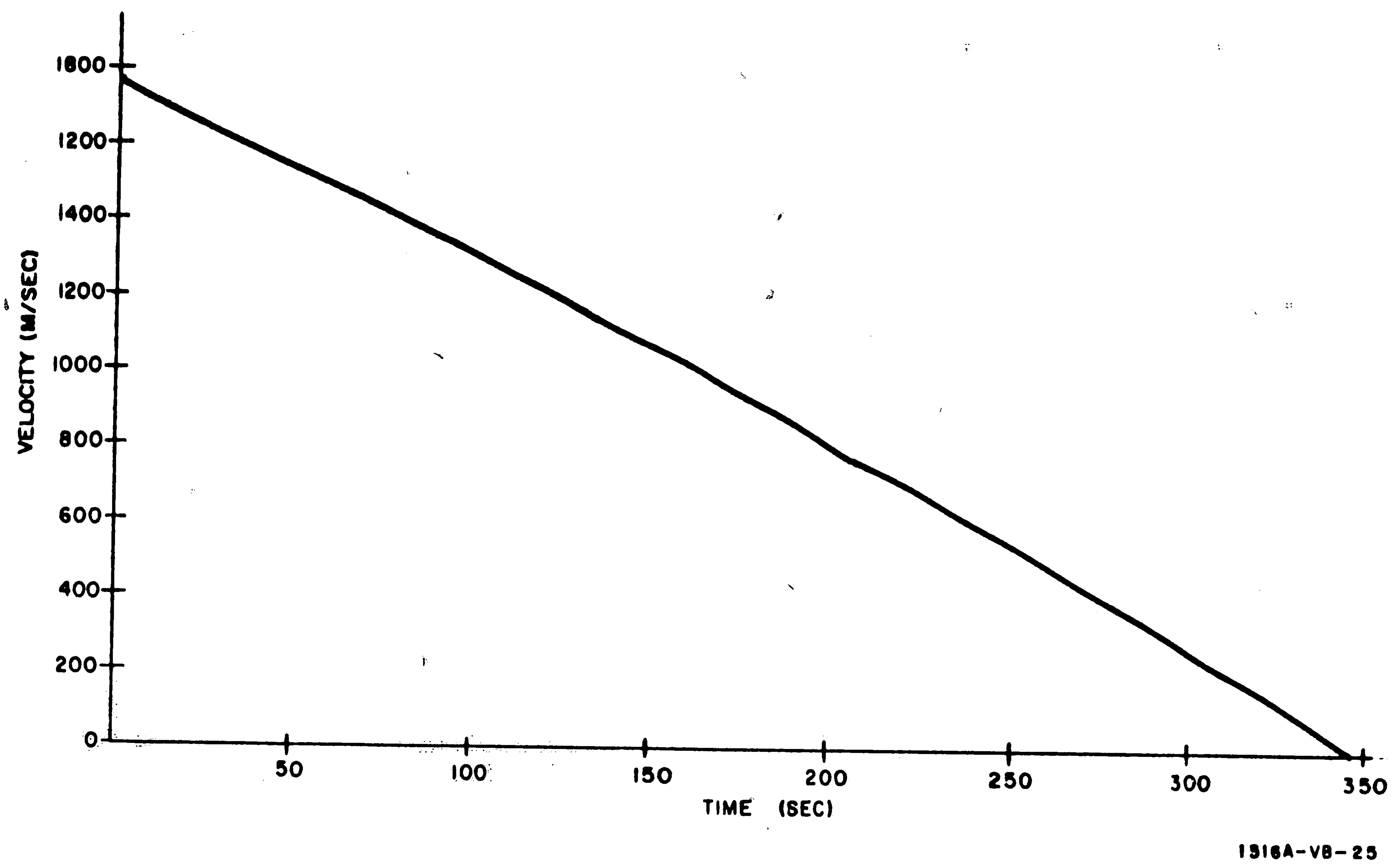

Figure 6. Reference Trajectory: Velocity vs. Time.

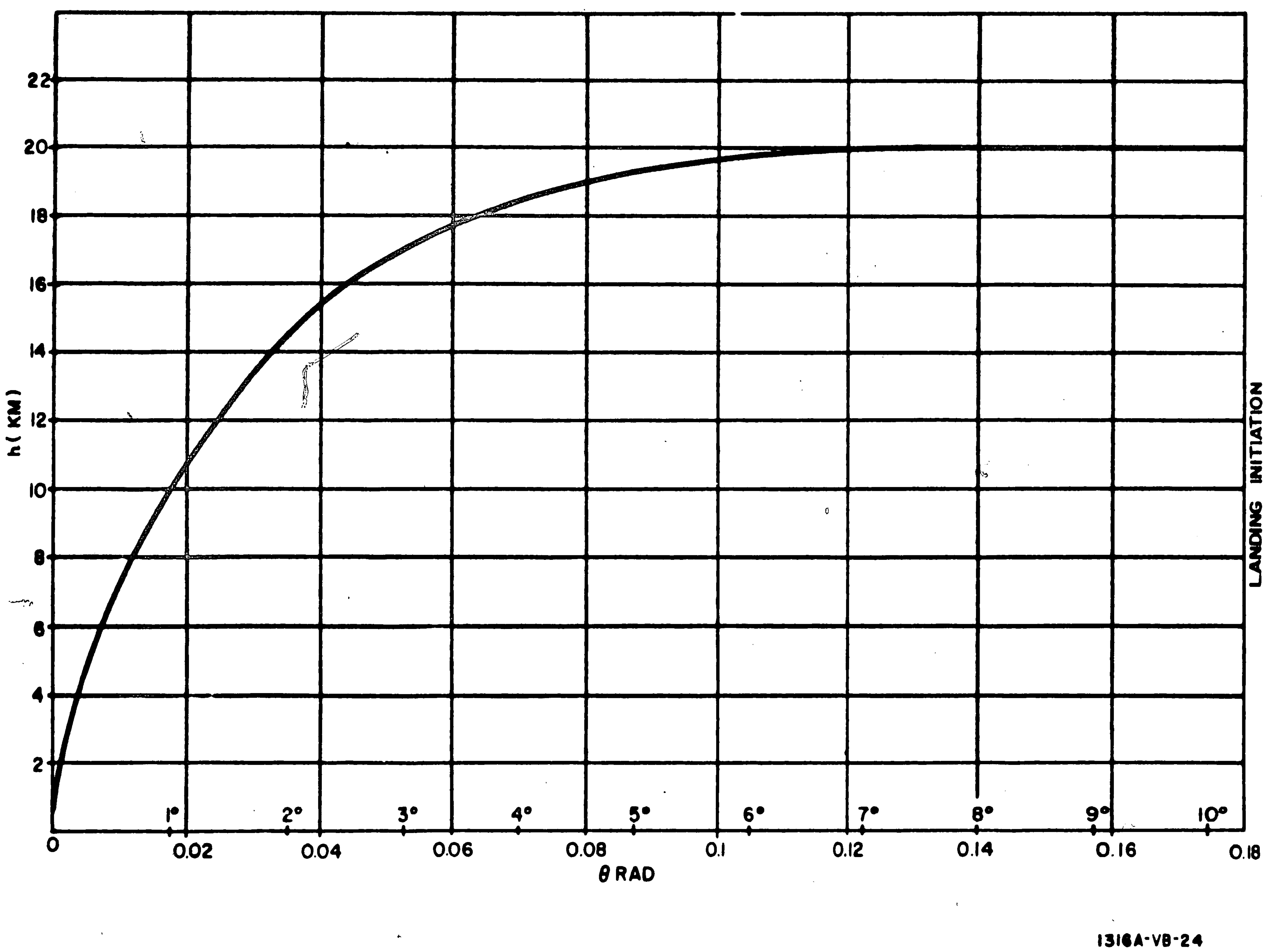

Figure 7. Reference Trajectory: Altitude (h) vs. Angular Displacement from Landing Site (θ).

For the system under study, the measured parameters are:

R: Line-of-sight range to a beacon located at the desired landing site.

$\dot{R}$: Line-of-sight range rate to the beacon.

ϕ: Line-of-sight angle to the beacon (measured from local vertical).

$\dot{\phi}$: Line-of-sight angular rate.

These quantities as well as the state variables are illustrated in figure 1.

It is shown in Appendix B that knowledge of R, $\dot{R}$, ϕ, and $\dot{\phi}$ is sufficient to allow determination of h, θ, γ, and V. Also in Appendix B, the required relationships between measured quantities and state variables are derived. These equations, which are given below, represent the estimated values of the state variables at the time of the measurement.

$$
\begin{aligned}
h &= R\cos\phi - r_o + \left(r_o^2 - R^2\sin^2\phi\right)^{1/2} \\
\theta &= \sin^{-1}\left[\frac{R}{r_o}\sin\phi\right] \\
\gamma &= \phi - \pi/2 + \tan^{-1}\left[\frac{R(\dot{\phi}+\dot{\theta})}{\dot{R}}\right] \\
V &= \left[\dot{R}^2 + R^2(\dot{\phi}+\dot{\theta})^2\right]^{1/2} \\
\dot{\theta} &= \frac{R\dot{\phi}\cos\phi + \dot{R}\sin\phi}{\left[r_o^2 - R^2\sin^2\phi\right]^{1/2}}
\end{aligned}
\qquad (2\text{-}4)
$$

3.0 ANALYTICAL APPROACH

With the preliminary analysis of section 2 concluded, it is possible to proceed to the actual error analysis which is the main object of this effort. Section 3.1 presents a description of the analytical approach and the techniques that have been used to determine the effects of sensor errors. Following that, section 3.2 covers the application of the analytical techniques to the guidance system selected for study.

3.1 Discussion of Available Analytical Techniques

The most accurate analysis would be a simulation of the guidance and control system. A multitude of simulated landings together with Monte Carlo statistical methods would be necessary to produce statistical estimates of sensor induced errors. To set up a simulation, two preliminary computer programs are required — one to determine reference trajectory parameters, and a second to compute the eight guidance terms (S_1, S_2, ..., S_8) as time functions along the reference flight path. Simulation of the landing phase would be considerably more complicated than the simulation of an orbital phase. The principal complication is introduced by the continuous thrust condition which makes it impossible to determine end point errors as functions of initial errors alone as can be done during free flight. It is also impossible to continually improve estimates of end conditions through data processing of redundant measurements as is done in many orbital guidance concepts.

As an alternative to a simulation analysis, linearization techniques are available. These are based on an assumption pertaining to the evaluation of partial derivatives in the neighborhood of a known point. Let,

$$y = f\,(x_1, x_2, \ldots, x_n) = f(x_i) \qquad (3\text{-}1)$$

where the x_i variables are implicit functions of time. If the reference value of y at some particular instant is y_o and the reference value of x_i is denoted by x_{io}, then

$$y_o = f(x_{io})$$

A linearized analysis is based on the assumption that small deviations of the x_i variables from the reference values will cause a small deviation in y which can be approximated by the expression

$$\delta y \equiv y - y_o \doteq \sum_{i=1}^{n} \left[\left(\frac{\partial y}{\partial x_i} \right)_{x_{io}} \right] \delta x_i \qquad (3\text{-}2)$$

$$\delta x_i \equiv x_i - x_{io}$$

Notice that the indicated partial derivatives are evaluated with the variables x_i taking on their reference values.

It was determined that an error analysis using linearization techniques would be much faster than a simulation approach. In addition, linearization is an extremely flexible approach, allowing changes to be made in sensor capabilities, trajectories, and observables with only minor alterations of the computer programs involved. The computer programs required are essentially the same two required as preliminary steps in a simulation analysis. Thus, the simpler, faster linearized analysis is a logical and valuable step in the development of an efficient simulation analysis of lunar landing vehicle performance.

In the light of the preceding statements, a linearized analysis seemed most valuable and was pursued.

3.2 Analysis

The analysis divides conveniently into two parts. The first is concerned with the propagation of observation errors through the guidance system to the thrust correction commands. The second part of the analysis relates to the effect of thrust command errors and thrust mechanization errors on final values of position and velocity.

3.2.1 Propagation of Observation Errors Through the Guidance System

The first step in the linearized analysis is to develop relationships between the instantaneous thrust command errors, and the observation errors which cause them. The thrust correction commands are calculated from the computed state variables, the stored state variables, and the stored guidance terms S_1 through S_8. The correction commands are computed from the following equations:

$$\delta T = S_1\ \delta h + S_2\ \delta\theta + S_3\ \delta\gamma + S_4\ \delta V \qquad (3\text{-}3)$$

$$\delta\alpha = S_5\ \delta h + S_6\ \delta\theta + S_7\ \delta\gamma + S_8\ \delta V$$

where

$$\begin{aligned} \delta h &\equiv h - h_r \\ \delta\theta &\equiv \theta - \theta_r \\ \delta\gamma &\equiv \gamma - \gamma_r \\ \delta V &\equiv V - V_r \end{aligned} \qquad (3\text{-}4)$$

The subscript r denotes reference values which are stored in the guidance computer as functions of time. All of the state variables are implicit functions of time. The computed value of a state variable is made up of two components - - the actual value, denoted by the subscript a and an error term denoted by e_h, e_θ, e_γ, and e_V with the subscript referring to the corresponding state variable. Then:

$$\begin{aligned} h &= h_a + e_h \\ \theta &= \theta_a + e_\theta \\ \gamma &= \gamma_a + e_\gamma \\ V &= V_a + e_V \end{aligned} \qquad (3\text{-}5)$$

It follows that:

$$
\begin{aligned}
\delta h &= \delta h_a + e_h = h_a - h_r + e_h \\
\delta \Theta &= \delta \Theta_a + e_\Theta \\
\delta \gamma &= \delta \gamma_a + e_\gamma \\
\delta V &= \delta V_a + e_V
\end{aligned}
\tag{3-6}
$$

The computed thrust deviation commands can be written as:

$$
\begin{aligned}
\delta T = \delta T_a + e_{Tc} = S_1 \left[\delta h_a + e_h\right] &+ S_2 \left[\delta \Theta_a + e_\Theta\right] \\
+ S_3 \left[\delta \gamma_a + e_\gamma\right] &+ S_4 \left[\delta V_a + e_V\right]
\end{aligned}
\tag{3-7}
$$

$$
\begin{aligned}
\delta \alpha = \delta \alpha_a + e_{\alpha c} = S_5 \left[\delta h_a + e_h\right] &+ S_6 \left[\delta \Theta_a + e_\Theta\right] \\
+ S_7 \left[\delta \gamma_a + e_\gamma\right] &+ S_8 \left[\delta V_a + e_V\right]
\end{aligned}
$$

where:

$e_{Tc} \equiv$ computed error in the thrust magnitude command

$e_{\alpha c} \equiv$ computed error in the thrust orientation command

$$
\begin{aligned}
e_{Tc} &= S_1 e_h + S_2 e_\Theta + S_3 e_\gamma + S_4 e_V \\
e_{\alpha c} &= S_5 e_h + S_6 e_\Theta + S_7 e_\gamma + S_8 e_V
\end{aligned}
\tag{3-8}
$$

If h, Θ, γ, and V were measured directly, the expressions for e_{Tc} and $e_{\alpha c}$ would complete the first part of the analysis. In general, however, these quantities cannot be observed directly, but rather, must be computed from other observed variables. For this analysis these observed quantities are R, $\dot{R}$, ϕ, and $\dot{\phi}$ as defined in section 2.5. Before determining expressions for e_h, e_Θ, e_γ, and e_V it is first necessary to relate the state variables to the observed information.

$$h = F_1 (R, \dot{R}, \phi, \dot{\phi})$$
$$\theta = F_2 (R, \dot{R}, \phi, \dot{\phi})$$
$$\gamma = F_3 (R, \dot{R}, \phi, \dot{\phi})$$
$$V = F_4 (R, \dot{R}, \phi, \dot{\phi})$$

The functions F_1, F_2, F_3, and F_4 derived in Appendix B are given in section 2.5. They are restated below for convenience.

$$
\begin{aligned}
h &= R \cos \phi - r_o + (r_o^2 - R^2 \sin^2 \phi)^{1/2} \\
\theta &= \sin^{-1} \left[\frac{R}{r_o} \sin \phi \right] \\
\gamma &= \phi - \frac{\pi}{2} + \tan^{-1} \left[\frac{R(\dot{\phi} + \dot{\theta})}{\dot{R}} \right] \\
V &= \left[\dot{R}^2 + R^2 (\dot{\phi} + \dot{\theta})^2 \right]^{1/2}
\end{aligned}
\qquad (3\text{-}9)
$$

where

$$\dot{\theta} = \frac{R \dot{\phi} \cos \phi + \dot{R} \sin \phi}{\left[r_o^2 - R^2 \sin^2 \phi \right]^{1/2}}$$

With these expressions available, it is possible to proceed to the next step which is the linearization process. It is desired to develop expressions relating errors in the computed values of the state variables to the observation errors which caused them. The application of linearization techniques yields these relationships in a small region near the reference trajectory.

$$
\begin{bmatrix} e_h \\ e_\theta \\ e_\gamma \\ e_V \end{bmatrix}
\doteq
\begin{bmatrix}
\frac{\partial F_1}{\partial R} & \frac{\partial F_1}{\partial \dot{R}} & \frac{\partial F_1}{\partial \phi} & \frac{\partial F_1}{\partial \dot{\phi}} \\
\frac{\partial F_2}{\partial R} & \frac{\partial F_2}{\partial \dot{R}} & \frac{\partial F_2}{\partial \phi} & \frac{\partial F_2}{\partial \dot{\phi}} \\
\frac{\partial F_3}{\partial R} & \frac{\partial F_3}{\partial \dot{R}} & \frac{\partial F_3}{\partial \phi} & \frac{\partial F_3}{\partial \dot{\phi}} \\
\frac{\partial F_4}{\partial R} & \frac{\partial F_4}{\partial \dot{R}} & \frac{\partial F_4}{\partial \phi} & \frac{\partial F_4}{\partial \dot{\phi}}
\end{bmatrix}
\times
\begin{bmatrix} e_R \\ e_{\dot{R}} \\ e_\phi \\ e_{\dot{\phi}} \end{bmatrix}
\qquad (3\text{-}10)
$$

All the partial derivatives are evaluated on the reference trajectory. The equations for the 16 partial derivatives are given in Appendix B.

The thrust correction command errors can then be approximated by the matrix equation:

$$\begin{bmatrix} e_{Tc} \\ \\ e_{\alpha c} \end{bmatrix} \doteq \begin{bmatrix} S_1 & S_2 & S_3 & S_4 \\ \\ S_5 & S_6 & S_7 & S_8 \end{bmatrix} \times \begin{bmatrix} e_h \\ e_\theta \\ e_\gamma \\ e_V \end{bmatrix} \qquad (3\text{-}11)$$

The S elements are defined in Appendix A. Substituting equation 3-10 into 3-11 and performing the indicated matrix multiplication yields the matrix:

$$\begin{bmatrix} e_{Tc} \\ \\ e_{\alpha c} \end{bmatrix} \doteq \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ \\ K_5 & K_6 & K_7 & K_8 \end{bmatrix} \times \begin{bmatrix} e_R \\ e_{\dot{R}} \\ e_\phi \\ e_{\dot{\phi}} \end{bmatrix} \qquad (3\text{-}12)$$

The Observation errors are assumed to be from independent populations displaying guassian statistics. The mean values of e_R, $e_{\dot{R}}$, e_ϕ, and $e_{\dot{\phi}}$ are all assumed to be zero; and the mean squared values are σ_R^2, $\sigma_{\dot{R}}^2$, σ_ϕ^2, and $\sigma_{\dot{\phi}}^2$ respectively. Multiplication of a gaussian distributed random variable by a constant results in a second gaussian distributed random variable. The mean and mean squared values of the new distribution are related to those of the old distribution by the constant multiplier and its square respectively. Let e_1 be a gaussian random variable with mean value μ_1 and mean squared value σ_1^2. If

$$e_2 = Ke_1$$

then

$$\mu_2 = K\mu_1$$

and

$$\sigma_2^2 = K^2 \sigma_1^2$$

From equation (3-12)

$$e_{Tc} \doteq K_1 e_R + K_2 e_{\dot{R}} + K_3 e_\phi + K_4 e_{\dot{\phi}}$$

$$e_{\alpha c} \doteq K_5 e_R + K_6 e_{\dot{R}} + K_7 e_\phi + K_8 e_{\dot{\phi}} \tag{3-13}$$

Since the observation errors are assumed to be independent random variables with zero mean, the mean values of e_Tc and $e_{\alpha c}$ are also zero and the mean squared values, σ^2_{Tc} and $\sigma^2_{\alpha c}$ can be approximated by:

$$\sigma^2_{Tc} \doteq K_1^2 \sigma^2_R + K_2^2 \sigma^2_{\dot{R}} + K_3^2 \sigma^2_\phi + K_4^2 \sigma^2_{\dot{\phi}}$$

$$\sigma^2_{\alpha c} \doteq K_5^2 \sigma^2_R + K_6^2 \sigma^2_{\dot{R}} + K_7^2 \sigma^2_\phi + K_8^2 \sigma^2_{\dot{\phi}} \tag{3-14}$$

The thrust error statistical parameters σ^2_{Tc} and $\sigma^2_{\alpha c}$ will vary in magnitude during the landing maneuver, because the eight multiplying factors K_1 through K_8 and the statistical quantities describing the observation error distributions are functions that vary during the landing. The method of dealing with these time varying error distributions will be discussed in the next section.

3.2.2 Propagation of Thrust Errors Along the Landing Trajectory

The expressions generated in the last section by no means complete the error analysis. Determining the relationships between errors in the computed thrust commands and the observation errors is a necessary step toward the desired end point, but does not contribute much in itself. The desired goal is a linearized expression relating the mean squared values of the position and velocity errors at the end of the landing trajectory to the mean squared values of the observation error distributions, and the mean squared values of thrust errors introduced by the control system itself. These latter errors will be referred to as thrust mechanization errors, and denoted by:

e_{Tm} = Thrust magnitude mechanization error

$e_{\alpha m}$ = Thrust angle mechanization error

The observation errors manifest themselves as thrust command errors at the output of the guidance system. There are then two sources of thrusting errors, one related to sensors, and one related to control system mechanization. The total thrust error is equal to the sum of these two components.

$$\begin{aligned} e_T &= e_{Tc} + e_{Tm} \\ e_\alpha &= e_{\alpha c} + e_{\alpha m} \end{aligned} \tag{3-15}$$

Rewriting the expressions for e_{Tc} and $e_{\alpha c}$ (equations 3-13), including the time arguments which now become important yields

$$\begin{aligned} e_{Tc}(t) &\doteq K_1(t)e_R(t) + K_2(t)e_{\dot{R}}(t) + K_3(t)e_\varphi(t) + K_4(t)e_{\dot{\varphi}}(t) \\ e_{\alpha c}(t) &\doteq K_5(t)e_R(t) + K_6(t)e_{\dot{R}}(t) + K_7(t)e_\varphi(t) + K_8(t)e_{\dot{\varphi}}(t) \end{aligned} \tag{3-16}$$

Then the total thrust error expressions can be written as:

$$\begin{aligned} e_T(t) &= e_{Tc}(t) + e_{Tm}(t) \\ e_\alpha(t) &= e_{\alpha c}(t) + e_{\alpha m}(t) \end{aligned} \tag{3-17}$$

The time variability of coefficients K_1 through K_8 and the observation error statistics cause the thrust error distribution to exhibit time varying statistics. Recalling that all the random error inputs are assumed to be independent and gaussian with zero mean values, the mean squared values of the thrust error distributions may be expressed as functions of time

$$
\begin{aligned}
\sigma_{T}^{2}(t) &\doteq K_1^2(t)\,\sigma_R^2(t) + K_2^2(t)\,\sigma_{\dot{R}}^2(t) + K_3^2(t)\sigma_{\phi}^2(t) + K_4^2(t)\sigma_{\dot{\phi}}^2(t) + \sigma_{Tm}^2(t) \\
\sigma_{\alpha}^{2}(t) &\doteq K_5^2(t)\,\sigma_R^2(t) + K_6^2(t)\,\sigma_{\dot{R}}^2(t) + K_7^2(t)\sigma_{\phi}(t) + K_8^2(t)\sigma_{\dot{\phi}}^2(t) + \sigma_{\alpha m}^2(t)
\end{aligned}
\tag{3-18}
$$

These equations determine the instantaneous statistics of the thrust errors.

The procedure used to determine the final state error statistics can be described as follows. First, the landing trajectory is divided into small time segments which are numbered sequentially from zero to N. The i^{th} segment is Δt_i seconds long, and the sum of all the small time segments is equal to the total time required by the nominal landing maneuver. It is assumed that the thrust error components and their statistics are constant over each Δt_i time segment, but independent of the errors existing in any other time segment, Δt_j.

Next, linearized expressions relating the constant thrust errors within a time interval (Δt_i) to the state variable errors resulting at the end of the interval are described. The partial derivatives associated with the linearization process are evaluated on the reference trajectory.

Finally, assuming that the thrust errors in each small time interval are statistically independent of the errors in each other interval implies that the final state variable errors which are the sum of all the small errors occurring in each time interval will have mean squared values which can be estimated by the sum of the mean squared errors contributed in each interval.

The first step is to determine the position and velocity errors in a small time interval that are caused by thrust errors. Thrust errors manifest themselves directly as acceleration errors. Therefore, thrust errors e_T and e_α can be related directly to errors in $\dot{V}$, $\ddot{h}$, and $\ddot{\theta}$ which are denoted $e_{\dot{V}}$, $e_{\ddot{h}}$, and $e_{\ddot{\theta}}$ respectively. The relationships are obtained by linearization about the reference trajectory and can be expressed as:

$$
\begin{aligned}
e_{\dot{V}}(t) &\doteq \frac{\partial \dot{V}}{\partial T}(t)\, e_T(t) + \frac{\partial \dot{V}}{\partial \alpha}(t)\, e_\alpha(t) \\
e_{\ddot{h}}(t) &\doteq \frac{\partial \ddot{h}}{\partial T}(t)\, e_T(t) + \frac{\partial \ddot{h}}{\partial \alpha}(t)\, e_\alpha(t) \qquad (3\text{-}19) \\
e_{\ddot{\theta}}(t) &\doteq \frac{\partial \ddot{\theta}}{\partial T}(t)\, e_T(t) + \frac{\partial \ddot{\theta}}{\partial \alpha}(t)\, e_\alpha(t)
\end{aligned}
$$

The partial derivatives are evaluated on the reference trajectory. (Expressions for these partials are given in Appendix C). These acceleration errors are essentially constant over the small intervals of time (Δt_i) so that the position and velocity errors in the i^{th} interval are:

$$
\begin{aligned}
e_{Vi} &\doteq \left[\frac{\partial \dot{V}}{\partial T} e_T + \frac{\partial \dot{V}}{\partial \alpha} e_\alpha\right]_{t=t_i} \Delta t_i \\
e_{hi} &\doteq \left[\frac{\partial \ddot{h}}{\partial T} e_T + \frac{\partial \ddot{h}}{\partial \alpha} e_\alpha\right]_{t=t_i} \frac{(\Delta t_i)^2}{2} \qquad (3\text{-}20) \\
e_{\theta i} &\doteq \left[\frac{\partial \ddot{\theta}}{\partial T} e_T + \frac{\partial \ddot{\theta}}{\partial \alpha} e_\alpha\right]_{t=t_i} \frac{(\Delta t_i)^2}{2}
\end{aligned}
$$

where,

$$t_n \equiv \sum_{i=0}^{i=n-1} \Delta t_i$$

The final position and velocity errors are obtained by summing the errors in each interval:

$$
\begin{aligned}
e_{Vf} &= \sum_{i=0}^{i=N-1} e_{Vi} \doteq \sum_{i=0}^{i=N-1} \Delta t_i \left[\frac{\partial \dot{V}}{\partial T} e_T + \frac{\partial \dot{V}}{\partial \alpha} e_\alpha\right]_{t=t_i} \\
e_{hf} &= \sum_{i=0}^{i=N-1} e_{hi} \doteq \sum_{i=0}^{i=N-1} \frac{(\Delta t_i)^2}{2}\left[\frac{\partial \ddot{h}}{\partial T} e_T + \frac{\partial \ddot{h}}{\partial \alpha} e_\alpha\right]_{t=t_i} \qquad (3\text{-}21 \\
e_{\theta f} &= \sum_{i=0}^{i=N-1} e_{\theta i} \doteq \sum_{i=0}^{i=N-1} \frac{(\Delta t_i)^2}{2}\left[\frac{\partial \ddot{\theta}}{\partial T} e_T \quad \frac{\partial \ddot{\theta}}{\partial \alpha} e_\alpha\right]_{t=t_i}
\end{aligned}
$$

The functions, $e_{\dot{T}}(t)$ and $e_\alpha(t)$ can be expanded as per equations (3-17) to give functions relating final state errors to observation and thrust mechanization errors.

$$
\begin{aligned}
e_{vf} \doteq \sum_{i=0}^{i=N-1} \Delta t_i \Big\{ & \left[\frac{\partial \dot{v}}{\partial T} K_1 + \frac{\partial \dot{v}}{\partial \alpha} K_5\right]_{t=t_i} e_R(t_i) \\
& + \left[\frac{\partial \dot{v}}{\partial T} K_2 + \frac{\partial \dot{v}}{\partial \alpha} K_6\right]_{t=t_i} e_{\dot{R}}(t_i) \\
& + \left[\frac{\partial \dot{v}}{\partial T} K_3 + \frac{\partial \dot{v}}{\partial \alpha} K_7\right]_{t=t_i} e_\phi(t_i) \\
& + \left[\frac{\partial \dot{v}}{\partial T} K_4 + \frac{\partial \dot{v}}{\partial \alpha} K_8\right]_{t=t_i} e_{\dot{\phi}}(t_i) \\
& + \left[\frac{\partial \dot{v}}{\partial T} e_{Tm} + \frac{\partial \dot{v}}{\partial \alpha} e_{\alpha m}\right]_{t=t_i} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
e_{hf} \doteq \sum_{i=0}^{i=N-1} \frac{(\Delta t_i)^2}{2} \Big\{ & e_R(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_1 + \frac{\partial \ddot{h}}{\partial \alpha} K_5\right]_{t=t_i} \\
& + e_{\dot{R}}(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_2 + \frac{\partial \ddot{h}}{\partial \alpha} K_6\right]_{t=t_i} \\
& + e_\phi(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_3 + \frac{\partial \ddot{h}}{\partial \alpha} K_7\right]_{t=t_i} \\
& + e_{\dot{\phi}}(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_4 + \frac{\partial \ddot{h}}{\partial \alpha} K_8\right]_{t=t_i} \\
& + \left[\frac{\partial \ddot{h}}{\partial T} e_{Tm} + \frac{\partial \ddot{h}}{\partial \alpha} e_{\alpha m}\right]_{t=t_i} \Big\}
\end{aligned}
\qquad 3\text{-}22
$$

$$
\begin{aligned}
e_{\theta f} \doteq \sum_{i=0}^{i=N-1} \frac{(\Delta t_i)^2}{2} \Big\{ & e_R(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_1 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_5\right]_{t=t_i} \\
& + e_{\dot{R}}(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_2 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_6\right]_{t=t_i} \\
& + e_\phi(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_3 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_7\right]_{t=t_i} \\
& + e_{\dot{\phi}}(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_4 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_8\right]_{t=t_i} \\
& + \left[\frac{\partial \ddot{\theta}}{\partial T} e_{Tm} + \frac{\partial \ddot{\theta}}{\partial \alpha} e_{\alpha m}\right]_{t=t_i} \Big\}
\end{aligned}
$$

It is true that the mean squared value of a sum of independent random variables is the sum of the mean squared values of the individual random variables. In general, this may be expressed as follows

$$\sigma_y^2 = \overline{\left[\sum_{i=0}^{i=N-1} k_i \, x_i\right]^2} = \sum_{i=0}^{i=N-1} k_i^2 \, \sigma_{x_i}^2 \tag{3-23}$$

The bar denotes mean value.

Applying equation (3-23) to the set of equations (3-22) results in the following expressions for the mean squared values of the final position and velocity error distribtuions.

$$\begin{aligned}
\sigma_{Vf}^2 \doteq \sum_{i=0}^{i=N-1} (\Delta t_i)^2 \Bigg\{ & \sigma_R^2(t_i)\left[\frac{\partial \dot{V}}{\partial T} K_1 + \frac{\partial \dot{V}}{\partial \alpha} K_5\right]^2_{t=t_i} \\
& + \sigma_{\dot{R}}^2(t_i)\left[\frac{\partial \dot{V}}{\partial T} K_2 + \frac{\partial \dot{V}}{\partial \alpha} K_6\right]^2_{t=t_i} \\
& + \sigma_{\phi}^2(t_i)\left[\frac{\partial \dot{V}}{\partial T} K_3 + \frac{\partial \dot{V}}{\partial \alpha} K_7\right]^2_{t=t_i} \\
& + \sigma_{\dot{\phi}}^2(t_i)\left[\frac{\partial \dot{V}}{\partial T} K_4 + \frac{\partial \dot{V}}{\partial \alpha} K_8\right]^2_{t=t_i} \\
& + \sigma_{Tm}^2(t_i)\left[\frac{\partial \dot{V}}{\partial T}\right]^2_{t=t_i} + \sigma_{\alpha m}^2(t_i)\left[\frac{\partial \dot{V}}{\partial \alpha}\right]^2_{t=t_i} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
\sigma_{hf}^2 \doteq \sum_{i=0}^{i=N-1} \frac{(\Delta t_i)^4}{4} \Bigg\{ & \sigma_R^2(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_1 + \frac{\partial \ddot{h}}{\partial \alpha} K_5\right]^2_{t=t_i} \\
& + \sigma_{\dot{R}}^2(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_2 + \frac{\partial \ddot{h}}{\partial \alpha} K_6\right]^2_{t=t_i} \\
& + \sigma_{\phi}^2(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_3 + \frac{\partial \ddot{h}}{\partial \alpha} K_7\right]^2_{t=t_i} \\
& + \sigma_{\dot{\phi}}^2(t_i)\left[\frac{\partial \ddot{h}}{\partial T} K_4 + \frac{\partial \ddot{h}}{\partial \alpha} K_8\right]^2_{t=t_i} \\
& + \sigma_{Tm}^2(t_i)\left[\frac{\partial \ddot{h}}{\partial T}\right]^2_{t=t_i} + \sigma_{\alpha m}^2(t_i)\left[\frac{\partial \ddot{h}}{\partial \alpha}\right]^2_{t=t_i} \Bigg\}
\end{aligned} \tag{3-24}$$

$$\sigma^2_{\theta f} \doteq \sum_{i=0}^{i=N-1} \frac{(\Delta t_i)^4}{4} \Big\{ \sigma^2_R(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_1 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_5\right]^2_{t=t_i}$$

$$+ \sigma^2_{\dot{R}}(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_2 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_6\right]^2_{t=t_i}$$

$$+ \sigma^2_{\phi}(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_3 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_7\right]^2_{t=t_i}$$

$$+ \sigma^2_{\dot{\phi}}(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T} K_4 + \frac{\partial \ddot{\theta}}{\partial \alpha} K_8\right]^2_{t=t_i} \qquad \text{(3-24 Cont'd)}$$

$$+ \sigma^2_{Tm}(t_i)\left[\frac{\partial \ddot{\theta}}{\partial T}\right]^2_{t=t_i} + \sigma^2_{\alpha m}(t_i)\left[\frac{\partial \ddot{\theta}}{\partial \alpha}\right]^2_{t=t_i} \Big\}$$

If the mean squared values of the observation and mechanization errors are constant, so that these quantities can be removed from the summations, the following expressions result. Definitions of $(VP)_{ji}$, $(HP)_{ji}$, and $(TP)_{ji}$ can be inferred by comparison with equation (3-24).

$$\sigma^2_{vf} \doteq \sigma^2_R \sum_{i=0}^{N-1} (VP)_{1i} + \sigma^2_{\dot{R}} \sum_{i=0}^{N-1} (VP)_{2i} + \sigma^2_{\phi} \sum_{i=0}^{N-1} (VP)_{3i}$$

$$+ \sigma^2_{\dot{\phi}} \sum_{i=0}^{N-1} (VP)_{4i} + \sigma^2_{Tm} \sum_{i=0}^{N-1} (VP)_{5i} + \sigma^2_{\alpha m} \sum_{i=0}^{N-1} (VP)_{6i}$$

$$\sigma^2_{hf} \doteq \sigma^2_R \sum_{i=0}^{N-1} (HP)_{1i} + \sigma^2_{\dot{R}} \sum_{i=0}^{N-1} (HP)_{2i} + \sigma^2_{\phi} \sum_{i=0}^{N-1} (HP)_{3i}$$

$$+ \sigma^2_{\dot{\phi}} \sum_{i=0}^{N-1} (HP)_{4i} + \sigma^2_{Tm} \sum_{i=0}^{N-1} (HP)_{5i} + \sigma^2_{\alpha m} \sum_{i=0}^{N-1} (HP)_{6i} \qquad \text{(3-25)}$$

$$\sigma^2_{\theta f} \doteq \sigma^2_R \sum_{i=0}^{N-1} (TP)_{1i} + \sigma^2_{\dot{R}} \sum_{i=0}^{N-1} (TP)_{2i} + \sigma^2_{\phi} \sum_{i=0}^{N-1} (TP)_{3i}$$

$$+ \sigma^2_{\dot{\phi}} \sum_{i=0}^{N-1} (TP)_{4i} + \sigma^2_{Tm} \sum_{i=0}^{N-1} (TP)_{5i} + \sigma^2_{\alpha m} \sum_{i=0}^{N-1} (TP)_{6i}$$

Another common way of expressing observation error statistics is in terms of a percent of the quantity being observed. That is,

$$\sigma_x = \eta_x x \quad \text{where, } (\eta_x)\ (100) = \text{percent}$$

Then,

$$\sigma_x^2 = \eta_x^2\, x^2$$

Equations expressing final error mean squared values would then be written as,

$$\sigma_{vf}^2 \doteq \eta_R^2 \sum_{i=0}^{N-1} (VP)_{1i}\, R^2(t_i) + \eta_{\dot R}^2 \sum_{i=0}^{N-1} (VP)_{2i}\, \dot R^2(t_i) + \eta_\phi^2 \sum_{i=0}^{N-1} (VP)_{3i}\, \phi^2(t_i)$$
$$+ \eta_{\dot\phi}^2 \sum_{i=0}^{N-1} (VP)_{4i}\, \dot\phi^2(t_i) + \eta_{Tm}^2 \sum_{i=0}^{N-1} (VP)_{5i}\, T^2(t_i) + \eta_{\alpha m}^2 \sum_{i=0}^{N-1} (VP)_{6i}\, \alpha^2(t_i)$$

$$\sigma_{hf}^2 \doteq \eta_R^2 \sum_{i=0}^{N-1} (HP)_{1i}\, R^2(t_i) + \eta_{\dot R}^2 \sum_{i=0}^{N-1} (HP)_{2i}\, \dot R^2(t_i) + \eta_\phi^2 \sum_{i=0}^{N-1} (HP)_{3i}\, \phi^2(t_i)$$
$$+ \eta_{\dot\phi}^2 \sum_{i=0}^{N-1} (HP)_{4i}\, \dot\phi^2(t_i) + \eta_{Tm}^2 \sum_{i=0}^{N-1} (HP)_{5i}\, T^2(t_i) + \eta_{\alpha m}^2 \sum_{i=0}^{N-1} (HP)_{6i}\, \alpha^2(t_i) \qquad (3\text{-}26)$$

$$\sigma_{\theta f}^2 \doteq \eta_R^2 \sum_{i=0}^{N-1} (TP)_{1i}\, R^2(t_i) + \eta_{\dot R}^2 \sum_{i=0}^{N-1} (TP)_{2i}\, \dot R^2(t_i) + \eta_\phi^2 \sum_{i=0}^{N-1} (TP)_{3i}\, \phi^2(t_i)$$
$$+ \eta_{\dot\phi}^2 \sum_{i=0}^{N-1} (TP)_{4i}\, \dot\phi^2(t_i) + \eta_{Tm}^2 \sum_{i=0}^{N-1} (TP)_{5i}\, T^2(t_i) + \eta_{\alpha m}^2 \sum_{i=0}^{N-1} (TP)_{6i}\, \alpha^2(t_i)$$

Combinations of equations (3-25) and (3-26) are also possible when certain of the error sources display constant statistical parameters and others show rms errors which are a constant percent of the observed quantity.

The values of $(VP)_{ji}$, $(HP)_{ji}$, and $(TP)_{ji}$ can be inferred by comparison with equations (3 - 24). For example,

$$\left[(VP)_{j1}\right]_{j=1} = \left\{(\Delta t_i)\left[\frac{\partial \dot{V}}{\partial T} K_1 + \frac{\partial \dot{V}}{\partial \alpha} K_5\right]_{t=t_i}\right\}^2 \qquad (3-27)$$

For the given guidance technique, reference trajectory, and observed variables; the summations of equations (3 - 25) and (3 - 26); once computed, represent the linearized transformations, or sensitivity coefficients, relating the input errors (caused by imperfect observation or mechanization) and the resulting position and velocity uncertainties at the end of the landing trajectory. The principal digital computer program developed for the landing analysis determines the values of these summations. This program has been implemented on an IBM 7094 digital computer.

3.3 Discussion of the Discrete Time Steps Utilized in the Analytical Model

Perhaps the best approach to this topic is to review the logical steps that lead to the introduction of discrete time steps in the analytical model. For reference, a simplified block diagram of the overall guidance and control system is given in figure 8.

In the absence of any random disturbance in the system, the actual thrust vector would be equal to the reference thrust vector plus whatever deviations from reference thrust correspond to the actual position and velocity deviations from the reference trajectory. This is called the pseudo-reference thrust vector.

When random errors enter the system, the actual thrust vector will deviate from the pseudo-reference vector in a random manner. To establish an analytical model, the continuous random error component is approximated by a random square wave having the following characteristics.

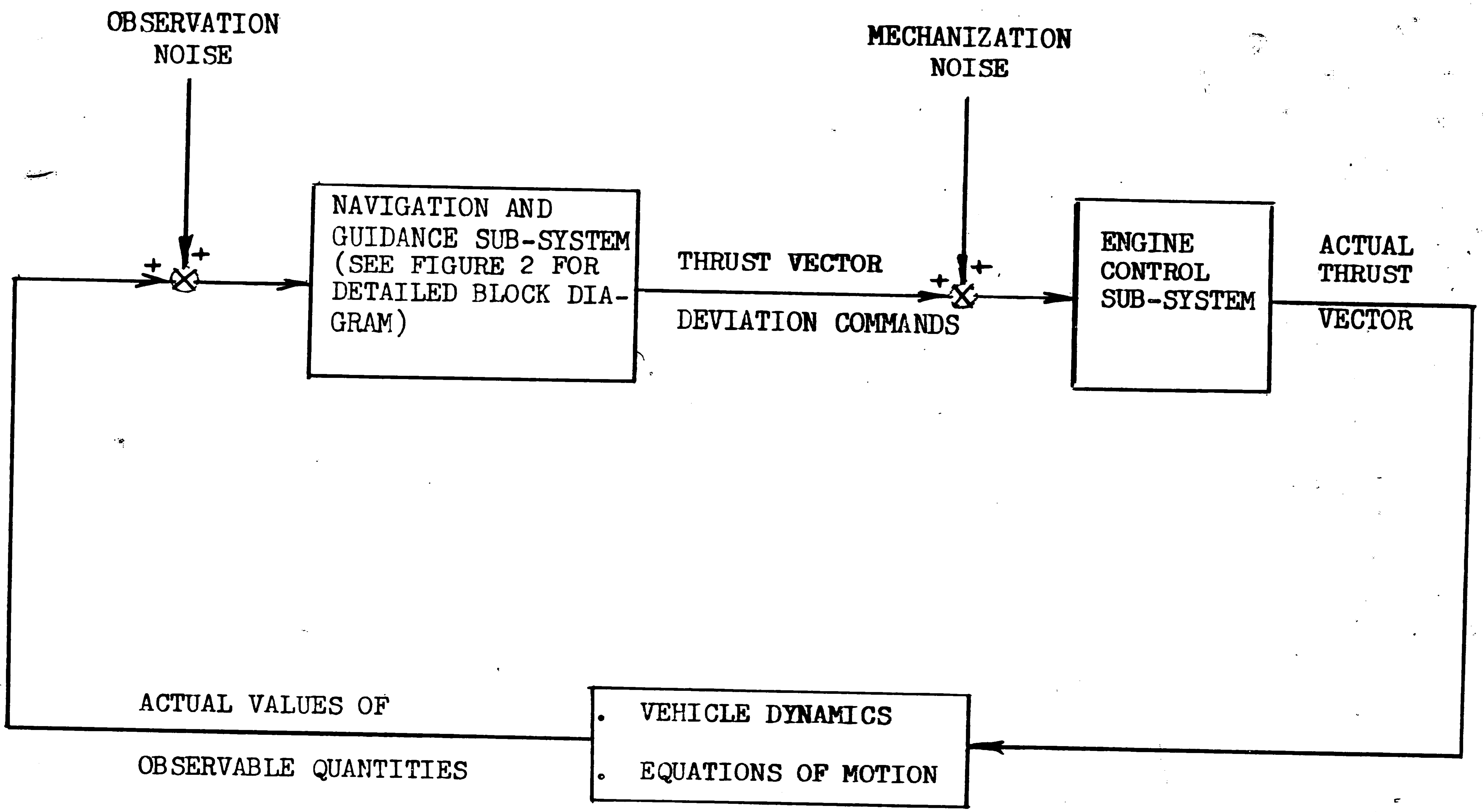

Figure 8. Overall Guidance and Control System Block Diagram.

- The random error is sampled at time t_i.
- The value at the sample time is held over Δt_i seconds. The quantity Δt_i is the thrust sampling interval.
- At $t_i + \Delta t_i$ the process is repeated.

The result is a signal that changes value periodically (every Δt_i seconds) and exhibits a constant, but random amplitude between sampling times. Throughout the actual analysis that has been performed, all of the small time intervals have been assumed to be of the same length so that the subscript "i" can be dropped.

In accordance with the preceding approximation, the entire trajectory is divided into small time intervals and a constant thrust vector error exists in each. The total position and velocity errors caused by sensor and mechanization noise are found by summing the errors occurring in each interval. Only statistical information is available concerning the random inputs so that statistical summation is implied above. The statistical problem is greatly reduced if the errors occurring in one interval are independent of errors in any other. This is equivalent to stating that the random thrust vector error in one time interval is uncorrelated with the random thrust error in any other.

This statement enables one to establish a relationship between the length of the analytical model time intervals (Δt) and the bandwidth of the engine control subsystem (see figure 8). In general, the correlation interval of white noise passed through a low pass filter of bandwidth B is approximately equal to 1/2B seconds. Therefore, if the error in successive time intervals are to be considered independent, Δt must be greater than $1/2B_1$, where B_1 is the engine control subsystem bandwidth.

In addition to the above statements concerning the length of the time intervals used in the analytical model there is an additional consideration. It is desired that the magnitudes of the random position and velocity errors which are caused by the thrust vector errors acting over the time Δt be considerably smaller than the sensor errors. A little thought reveals that this is equivalent to saying that the overall system should not be capable of responding completely to thrust errors before a new command is computed. Since the analytical model has some of the characteristics of a sampled data control system (thrust vector errors are sampled, held over a period of time, and then sampled again) it can be seen that this restriction is equivalent to the minimum sampling rate restriction required to insure the stability of a sampled data control system.

4.0 RESULTS

The primary results obtained from the computer program are the values of the error sensitivity coefficients represented by the summatiors derived in the previous section. These summations relate the mean squared values of the observation and mechanization errors to the mean squared errors in terminal position and velocity.

Let σ^2_{ij} denote the mean squared error in the final value of i^{th} state variable attributable to the j^{th} error source. The indices i and j correspond to state variables and error sources as given below.

Value of i	Corresponding State Variable
1	h: altitude
2	θ: angular displacement from landing site
3	γ: flight path angle
4	V: velocity vector magnitude
Value of j	**Corresponding Error Source**
1	R: observation of range
2	$\dot{R}$: observation of range rate
3	ϕ: angular measurement
4	$\dot{\phi}$: angular rate measurement
5	Thrust magnitude control system
6	Thrust direction control system

For example, σ^2_{14} is the mean squared value of final altitude error caused by errors in the measurement of $\dot{\phi}$. Since all the error sources are assumed to be independent, the total mean squared error in the terminal value of the i^{th} state variable is:

$$\sigma_i^2 = \sum_{j=1}^{6} \sigma_{ij}^2 \qquad (4\text{-}1)$$

If the statistics of each error source are constant,

$$\sigma_{ij}^2 = K_{ij}^2 \ \sigma_j^2$$

where σ_j^2 is the effective mean squared error contributed by the j^{th} error source and $(K_{ij})^2$ is the appropriate error sensitivity coefficient as determined from the computer program. If the rms error in the j^{th} error source is allowed to be a constant percentage of the observed or controlled quantity, a different set of equations result:

$$\sigma_{ij}^2 = (K'_{ij})^2 \eta_j^2$$

The multiplier $(K'_{ij})^2$ is a modified error sensitivity coefficient, and η^2_j is the constant error percentage (expressed as a decimal fraction of unity) squared. There are two options to the computer program--one computes the coefficients $(K_{ij})^2$ and the second is a composite program which computes $(K'_{i1})^2$, $(K'_{i2})^2$, $(K_{i3})^2$, $(K'_{i4})^2$, $(K_{i5})^2$, and $(K_{i6})^2$. That is, rms errors in range, range rate, and angle rate measurements are taken to be a constant percent of the measured quantities; while errors in angle measurement and thrust mechanization are allowed to have constant statistics.

For the group of observables sensed by the radar beacon tracker, it was felt that range and range rate measurements would be best represented as having a constant percent of error; while the remaining error sources would exhibit error distributions with nearly constant statistical parameters over the landing trajectory. Therefore, values of the error sensitivity coefficients $(K_{i1})^2$, $(K_{i2})^2$, $(K_{i3})^2$, $(K_{i4})^2$, $(K_{i5})^2$, $(K_{i6})^2$, $(K'_{i1})^2$, $(K'_{i2})^2$, and $(K'_{i4})^2$ have been computed and appear in table 1. The square root of each coefficient is also presented. This quantity relates the rms value of any particular final error component to the rms error, or percent error, of the appropriate error source.

Table 1: Computed Sensitivity Coefficients.

State Variable Index	Error Source Index	Δt_i = 1 sec.		Δt_i = 2 sec.		Δt_i = 3 sec.	
		$(K_{ij})^2$	(K_{ij})	$(K_{ij})^2$	(K_{ij})	$(K_{ij})^2$	(K_{ij})
i = 1	j = 1	1.38×10^{1}	3.72×10^{0}	1.35×10^{1}	3.68×10^{0}	1.46×10^{0}	1.21×10^{0}
	j = 2	6.31×10^{0}	2.52×10^{0}	7.02×10^{0}	2.63×10^{0}	5.93×10^{0}	2.44×10^{0}
	j = 3	4.56×10^{5}	6.76×10^{2}	4.40×10^{5}	6.64×10^{2}	1.03×10^{5}	3.21×10^{2}
	j = 4	1.88×10^{6}	1.37×10^{3}	3.59×10^{6}	1.87×10^{3}	5.29×10^{6}	2.30×10^{3}
	j = 5	4.30×10^{-6}	2.08×10^{-3}	8.58×10^{-6}	2.93×10^{-3}	1.28×10^{-5}	3.58×10^{-3}
	j = 6	0.0	0.0	0.0	0.0	0.0	0.0
i = 2	j = 1	1.49×10^{0}	1.22×10^{0}	1.88×10^{0}	1.37×10^{0}	1.80×10^{0}	1.34×10^{0}
	j = 2	5.01×10^{-1}	7.09×10^{-1}	1.67×10^{0}	1.29×10^{0}	3.09×10^{0}	$1.76c10^{0}$
	j = 3	3.54×10^{4}	1.88×10^{2}	5.52×10^{4}	2.36×10^{2}	7.41×10^{4}	2.73×10^{2}
	j = 4	8.34×10^{7}	9.14×10^{3}	6.70×10^{8}	2.59×10^{4}	2.27×10^{9}	4.77×10^{4}
	j = 5	9.59×10^{-8}	3.10×10^{-4}	7.49×10^{-7}	8.65×10^{-4}	2.40×10^{-6}	1.55×10^{-3}
	j = 6	2.26×10^{3}	4.76×10^{1}	1.81×10^{4}	1.35×10^{2}	6.10×10^{4}	2.48×10^{2}
i = 3	j = 1	7.40×10^{-2}	2.72×10^{-1}	1.15×10^{-1}	3.40×10^{-1}	1.49×10^{-1}	3.86×10^{-1}
	j = 2	5.30×10^{-1}	7.28×10^{-1}	2.31×10^{0}	1.52×10^{0}	5.35×10^{0}	2.31×10^{0}
	j = 3	5.33×10^{3}	7.30×10^{1}	1.01×10^{4}	1.00×10^{2}	1.62×10^{4}	1.28×10^{2}
	j = 4	1.34×10^{6}	1.16×10^{3}	9.53×10^{6}	3.09×10^{3}	3.06×10^{7}	5.54×10^{3}
	j = 5	9.64×10^{-7}	9.82×10^{-4}	7.71×10^{-6}	2.78×10^{-3}	2.60×10^{-5}	5.10×10^{-3}
	j = 6	2.21×10^{2}	1.49×10^{1}	1.73×10^{3}	4.16×10^{1}	5.52×10^{3}	7.44×10^{1}

State Variable Notation

- i = 1: Final velocity error component
- i = 2: Final altitude error component
- i = 3: Final horizontal range error component

Error Source Notation

- j = 1: Range observation
- j = 2: Range rate observation
- j = 3: Angle observation
- j = 4: Angle rate observation
- j = 5: Thrust magnitude mechanization
- j = 6: Thrust orientation mechanization

Table 1: (Continued)

State Variable Index	Error Source Index	$\Delta t_i = 1$ sec.		$\Delta t_i = 2$ sec.		$\Delta t_i = 3$ sec.	
		$(K'_{ij})^2$	(K'_{ij})	$(K'_{ij})^2$	(K'_{ij})	$(K'_{ij})^2$	(K'_{ij})
i = 1	j = 1	3.49×10^6	1.87×10^3	3.40×10^6	1.84×10^3	4.05×10^5	6.36×10^2
	j = 2	3.41×10^4	1.85×10^2	6.82×10^4	2.61×10^2	1.05×10^5	3.20×10^2
	j = 3	n.c*.	n.c.	n.c.	n.c.	n.c.	n.c.
	j = 4	1.30×10^2	1.14×10^1	2.56×10^2	1.60×10^1	3.68×10^2	1.92×10^1
	j = 5	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.
	j = 6	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.
i = 2	j = 1	3.80×10^5	6.16×10^2	5.07×10^5	7.12×10^2	5.52×10^5	7.43×10^2
	j = 2	4.74×10^2	2.18×10^1	3.74×10^3	6.11×10^1	1.23×10^4	1.11×10^2
	j = 3	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.
	j = 4	1.22×10^2	1.10×10^1	9.77×10^2	3.12×10^1	3.30×10^3	5.74×10^1
	j = 5	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.
	j = 6	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.
i = 3	j = 1	2.06×10^4	1.44×10^2	4.51×10^4	2.12×10^2	9.04×10^4	3.05×10^2
	j = 2	8.55×10^3	9.24×10^1	6.84×10^4	2.62×10^2	2.31×10^5	4.81×10^2
	j = 3	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.
	j = 4	1.62×10^2	1.28×10^1	6.84×10^3	3.57×10^1	5.05×10^3	6.36×10^1
	j = 5	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.
	j = 6	n.c.	n.c.	n.c.	n.c.	n.c.	n.c.

$(K_{ij})^2$ relates final errors to error sources with constant statistical parameters.

$(K'_{ij})^2$ relates final errors to error sources with rms error expressed as a percentage of the observable

* n.c: not computed

For the computer program, all of the small time intervals (Δt_i) were allowed to have equal length. The program was run with Δt_i equal to one, two, and three seconds.

To examine the effects of sensor and mechanization errors on the terminal position and velocity errors, the sensitivity coefficients were used to generate the curves shown in figure 9 through 14. These figures show the final position and velocity errors (rms) plotted against the rms observation and mechanization errors. As mentioned above, a composite of K_{ij} and K'_{ij} coefficients best represents the lunar landing situation. Figure 9 shows the rms values of final position and velocity errors resulting from range observation errors along the trajectory. The rms value of the range observation errors is equal to η_1 times 100 percent of the range measurement at any point in the trajectory. As a typical example, if η_1 equals 0.01 (1%) the resulting position and velocity error components are:

Velocity error (rms) = 18.5 meters/second

Altitude error (rms) = 6.2 meters

Horizontal range error (rms) = 1.44 meters

Similarly, through application of curves 10 through 14 the final error components produced by the other five error sources can be estimated. Of the remaining five sources, only one, the range rate observation has statistical parameters that vary along the trajectory. The range rates that will occur generally fall well within the dynamic range of range rate detection techniques. The observation accuracy is then limited by non-linearities which produce a constant percent of error. The angular rate observation errors are allowed to be from a time invariant population because the angular rates encountered during the trajectory are near the lower end of the dynamic range of angle rate measuring techniques. Thrust magnitude errors can be expressed either in terms of percent error or in the units of thrust since the thrust

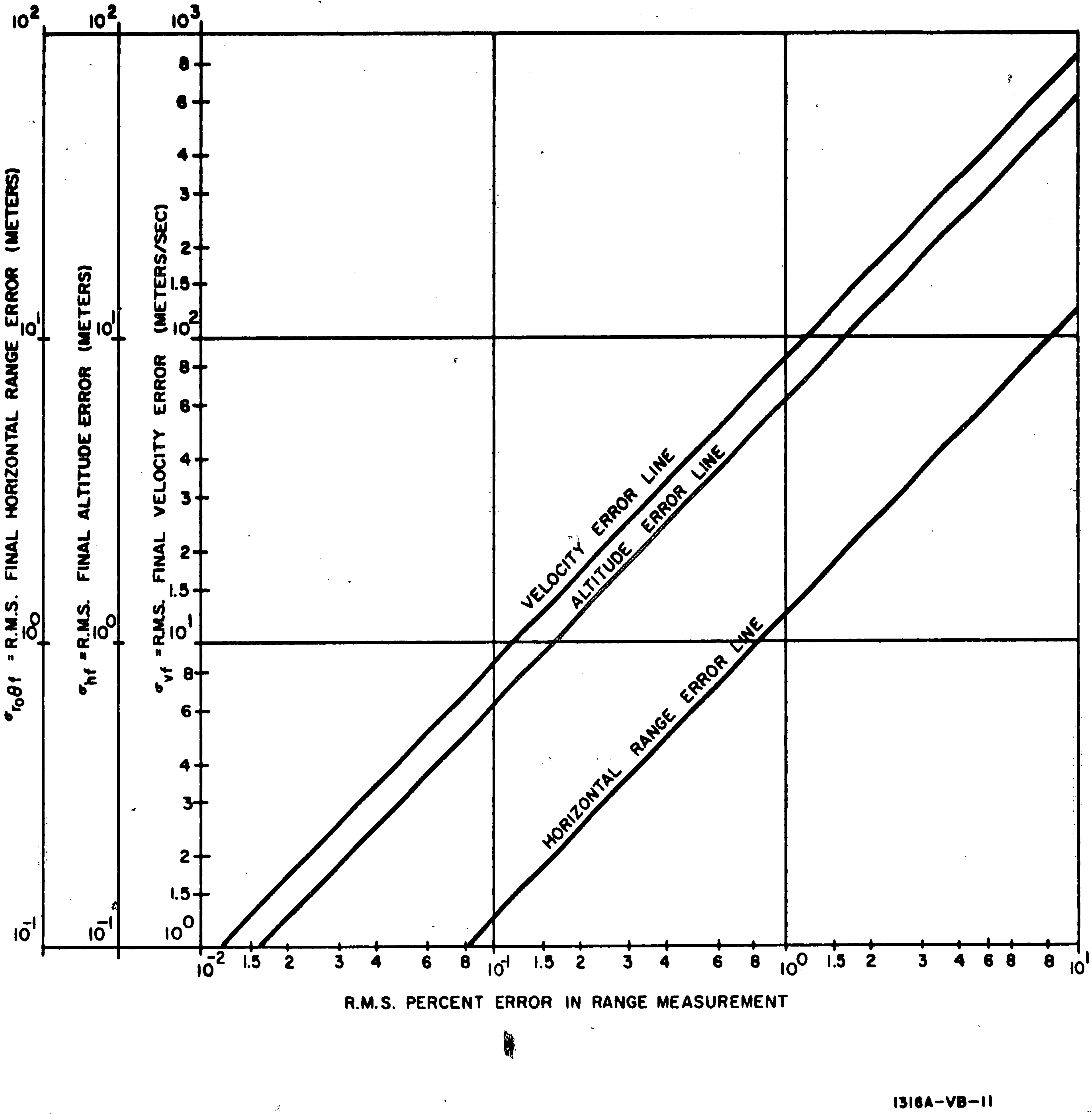

Figure 9. Final Error Components Attributable to Range Measurement Errors.

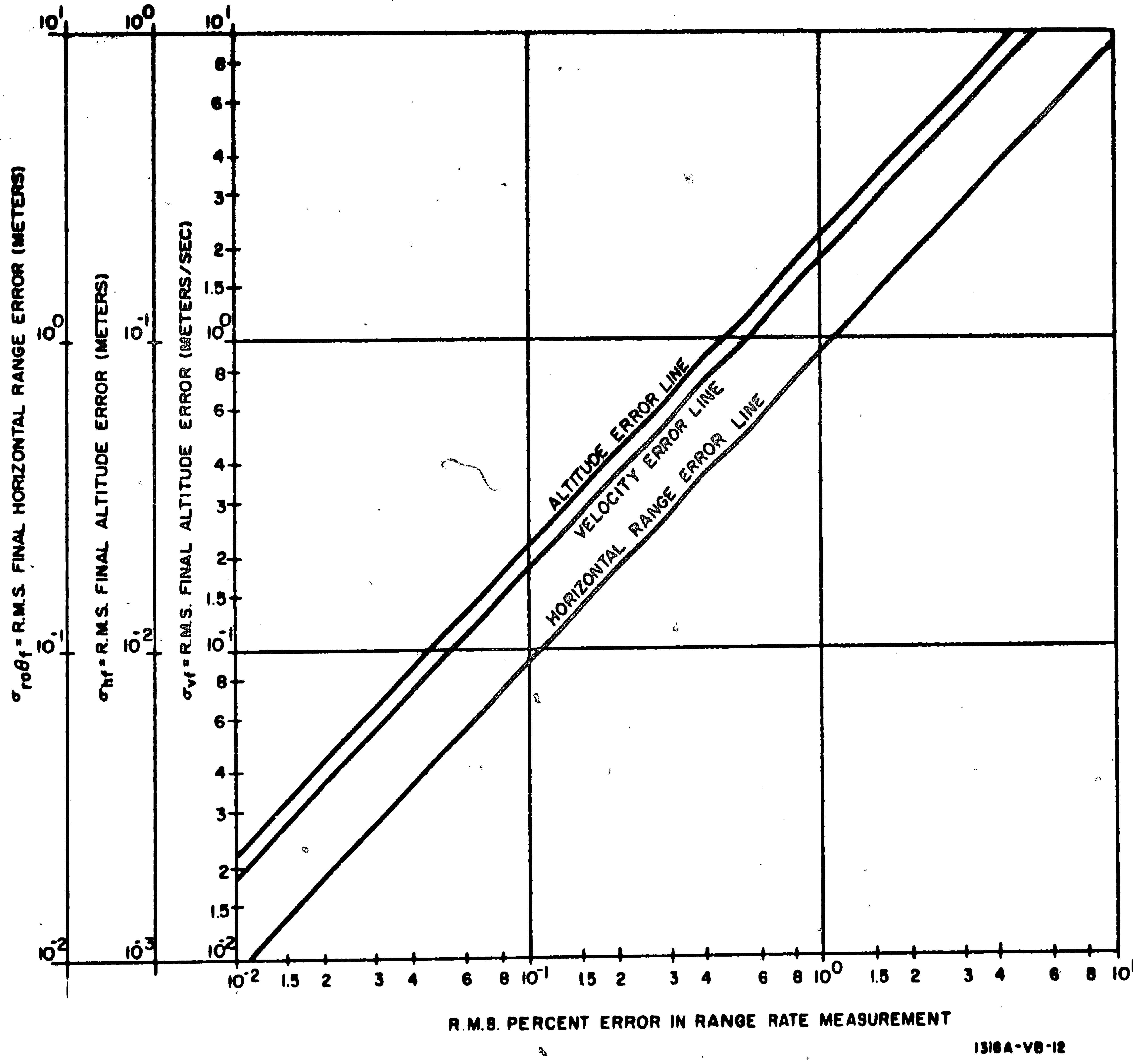

Figure 10. Final Error Components Attributable to Range Rate Measurement Errors.

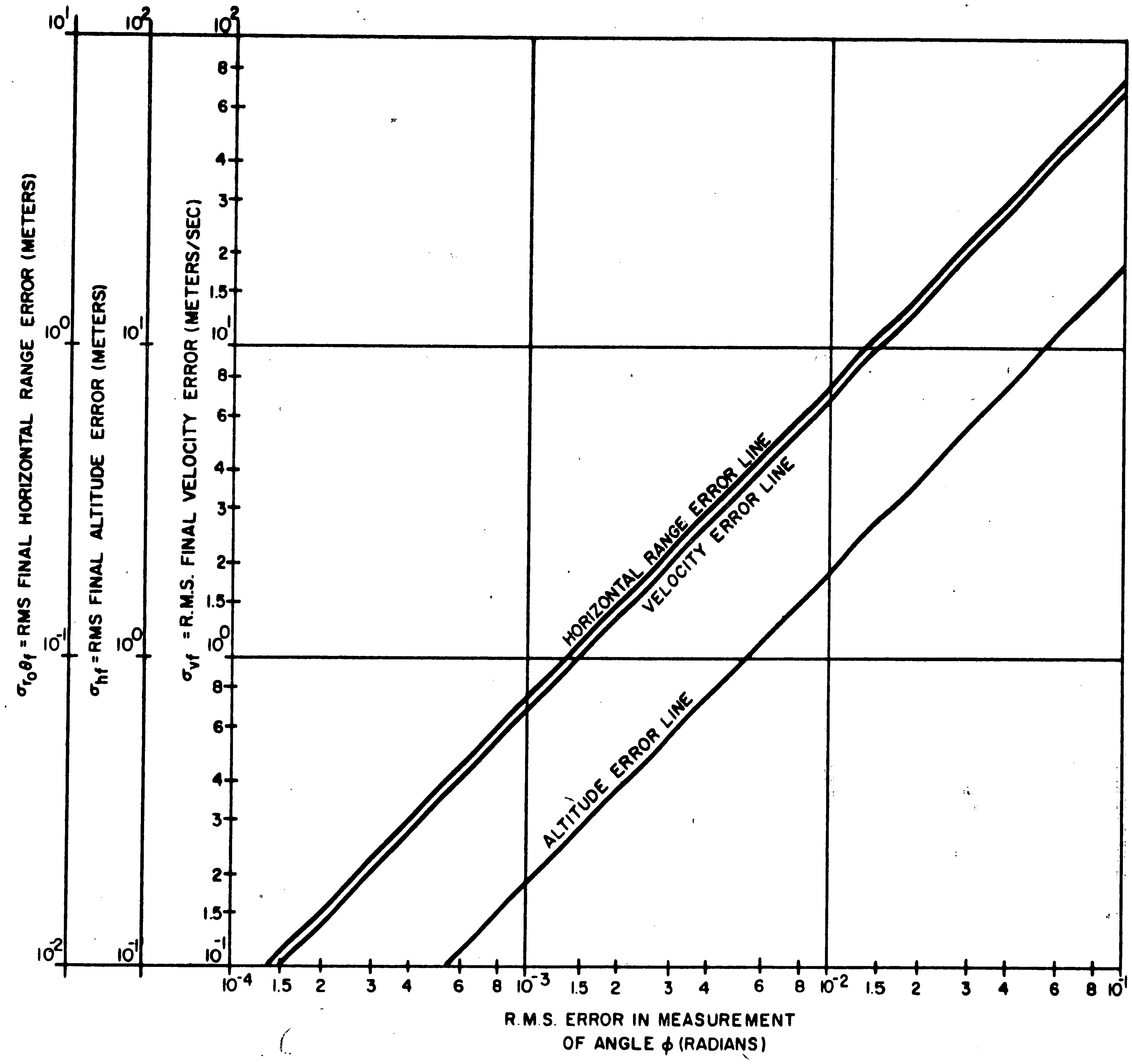

Figure 11. Final Error Components Attributable to Angle Measurement Errors.

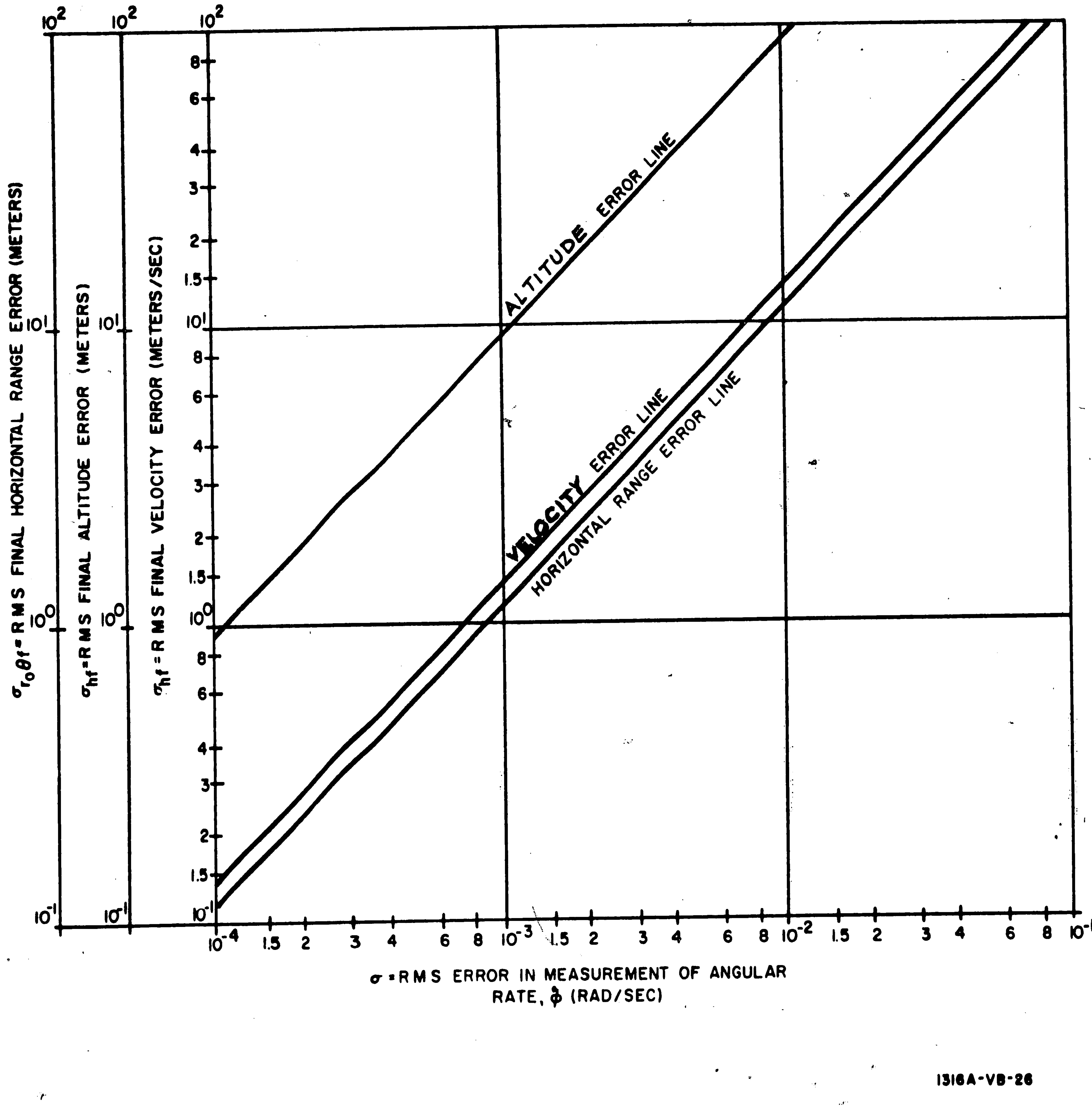

Figure 12. Final Error Components Attributable to Angle Rate Measurement Errors.

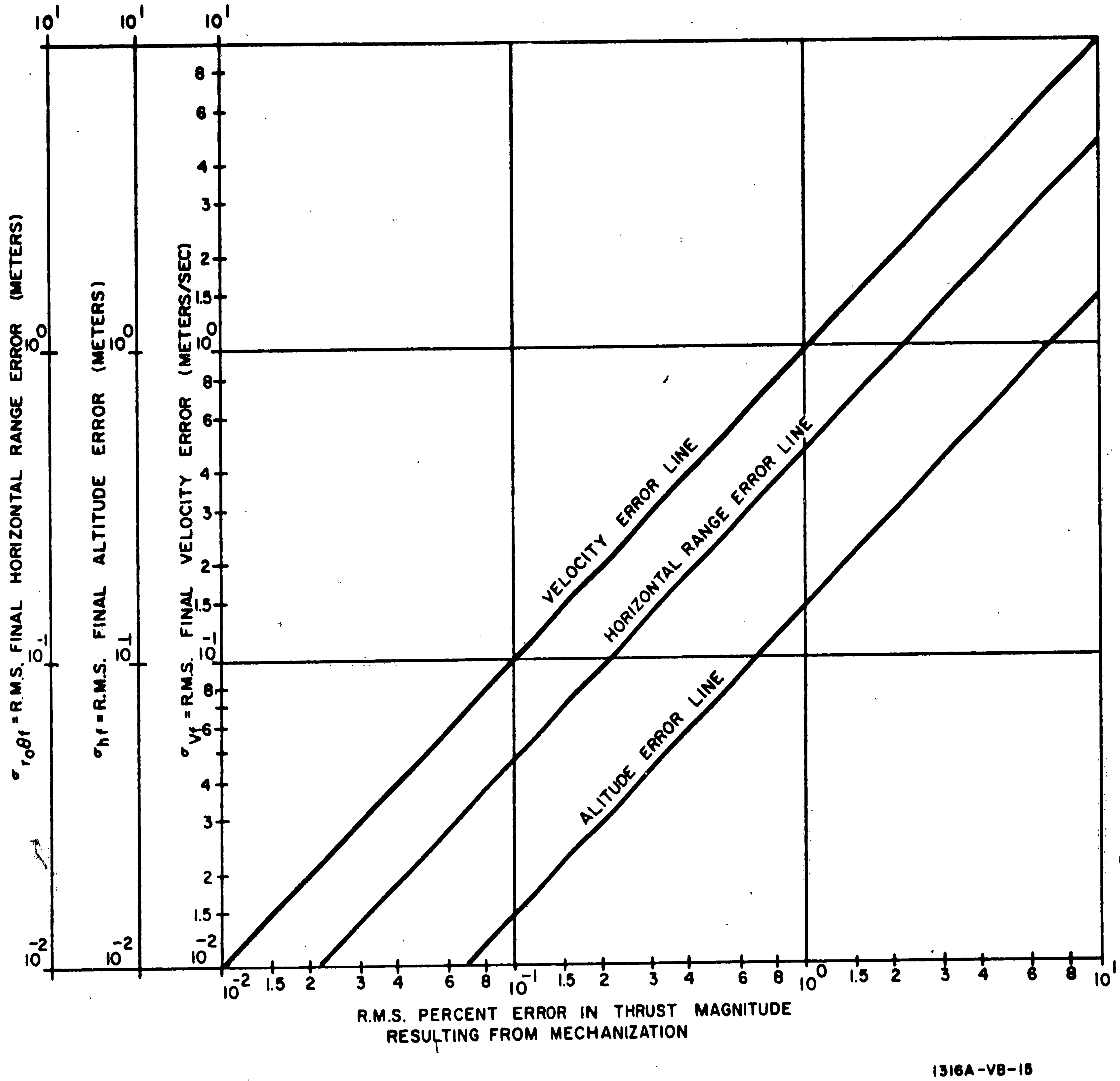

Figure 13. Final Error Components Attributable to Thrust Magnitude Mechanization Errors.

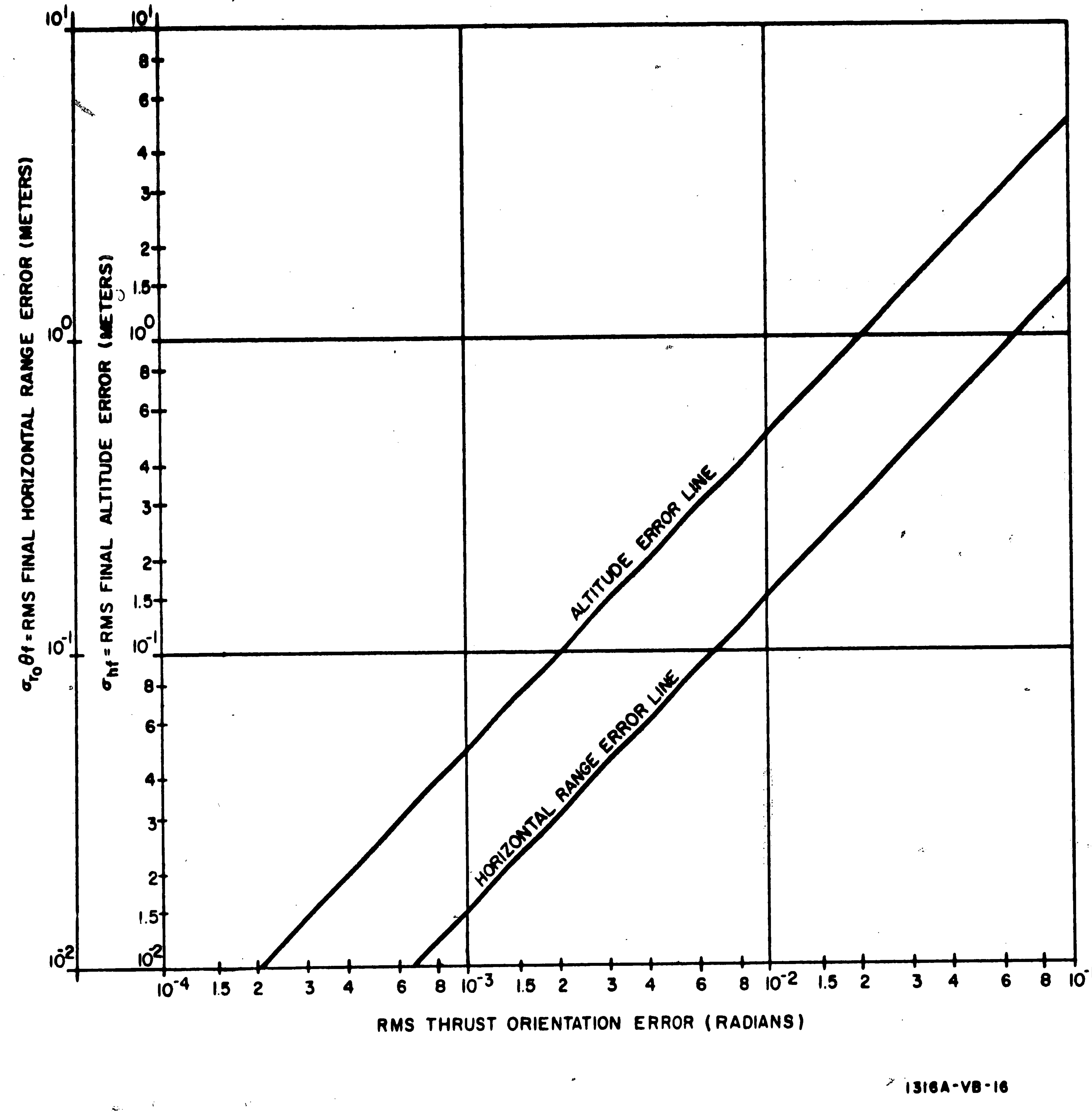

Figure 14. Final Error Components Attributable to Thrust Orientation Mechanization Errors.

is essentially constant throughout the maneuver. Angle measurement errors, are hardly ever expressed in terms of a percent of the angle measured, so that it was not deemed necessary to consider this representation for angular error sources.

If numbers which are considered to be typical of the present state-of-the-art in sensor capability are assumed it is possible to generate estimates of the final position and velocity errors would result by using the error sensitivity coefficients. This will be done as a sample calculation. Let the error sources have the following rms values.

Range observation: 0.1% of range

Range rate observation: 0.5% of range rate

Angle observation: 1 milli-radian

Angle rate observation: 1 milli-radian/second

Thrust magnitude mechanization: 1% = 400 newtons

Thrust angle mechanization: 35 milli-radians

For these sample radar error parameters, the rms values of the final position and velocity error components have been computed and appear in table 2. (Table 2 is compiled for Δt_i equal to one second.)

It should be recalled that errors described in table 2 are deviations from the end conditions which would result in the absence of any control and/or sensor uncertainties. They do not necessarily represent deviations from the reference end point as discussed in section 2.3.

Table 2: Results of Sample Computation.

A) rms velocity errors (meters/sec.)	
1) caused by range observation errors	1.87
2) caused by range rate observation errors	0.93
3) caused by angle observation errors	0.68
4) caused by angle rate observation errors	1.37
5) caused by thrust magnitude errors	1.00
6) caused by thrust orientation errors*	0.0
TOTAL RMS VELOCITY ERROR (METERS/SEC)	2.78
B) rms altitude errors (meters)	
1) caused by range observation errors	0.62
2) caused by range rate observation errors	0.11
3) caused by angle observation errors	0.19
4) caused by angle rate observation errors	9.14
5) caused by thrust magnitude errors	0.15
6) caused by thrust orientation errors	1.67
TOTAL RMS ALTITUDE ERROR (METERS)	9.40
C) rms horizontal range errors (meters)	
1) caused by range observation errors	0.14
2) caused by range rate observation errors	0.46
3) caused by angle observation errors	0.07
4) caused by angle rate observation errors	1.16
5) caused by thrust magnitude errors	0.48
6) caused by thrust orientation errors	0.52
TOTAL HORIZONTAL RANGE ERROR, RMS (M)	1.44

*Linearization about the reference trajectory results in $K_{16} = 0$.

5.0 APPENDIX A: GUIDANCE LAW DERIVATION

The guidance terms are derived by following the procedure described in reference 2. The following presentation carries through to completion the derivation of guidance terms S_1 through S_8 with time as the independent variable which is outlined in this reference.

The applicable equations of motion are:

$$\begin{aligned}
\dot{h} &= V \sin \gamma \\
\dot{\theta} &= -\frac{V \cos \gamma}{r_o + h} \\
\dot{\gamma} &= \frac{V \cos \gamma}{r_o + h} - \frac{T \sin \alpha}{m V} - \frac{\mu \cos \gamma}{V (r_o + h)^2} \\
\dot{V} &= -\frac{T \cos \alpha}{m} - \frac{\mu \sin \gamma}{(r_o + h)^2}
\end{aligned} \qquad (5-1)$$

Where:

$V \equiv$ magnitude of the velocity vector;

$r_o \equiv$ radius of reference lunar sphere;

$T \equiv$ magnitude of applied thrust vector;

$m \equiv$ mass of landing vehicle;

$\mu \equiv$ lunar gravitational constant.

The remaining parameters are defined in figure 1. For a constant thrust, gravity turn trajectory,

$T = T_o$ = a constant

and

$\alpha = 0$

on the nominal trajectory.

The perturbation equations relating deviations in the dynamic state variable derivatives to deviations in state variables and thrust control quantities, in matrix form, are:

$$\left[\delta\dot{\xi}_i\right] \doteq \left[\frac{\partial\dot{\xi}_i}{\partial\xi_j}\right]\left[\delta\xi_j\right] + \left[\frac{\partial\dot{\xi}_i}{\partial\nu_j}\right]\left[\delta\nu_j\right] \tag{5-2}$$

For ease of handling, the state variables and control quantities have been redefined.

$$\dot{\xi}_i \equiv (\dot{h})_{i=1} \;;\; (\dot{\theta})_{i=2} \;;\; (\dot{\gamma})_{i=3} \;;\; (\dot{V})_{i=4}$$

$$\xi_j \equiv (h)_{j=1} \;;\; (\theta)_{j=2} \;;\; (\gamma)_{j=3} \;;\; (V)_{j=4}$$

$$\nu_j \equiv (T)_{j=1} \;;\; (\alpha)_{j=2}$$

For example, $\frac{\partial\dot{\xi}_3}{\partial\xi_2} = \frac{\partial\dot{\gamma}}{\partial\theta}$, and $\frac{\partial\dot{\xi}_2}{\partial\nu_2} = \frac{\partial\dot{\theta}}{\partial\alpha}$.

The operator, δ, indicates a deviation from the reference value.

The equations adjoint to the perturbation equations are:

$$\left[\dot{\lambda}_i\right] = -\left[\frac{\partial\dot{\xi}_i}{\partial\xi_j}\right]^T\left[\lambda_j\right] \tag{5-3}$$

for i = 1, 2, 3, 4; and j = 1, 2, 3, 4. The super script "T" indicates the transpose matrix.

If equation (5-2) is pre-multiplied by $\left[\lambda_j\right]^T$, and (5-3) by $\left[\delta\xi_j\right]^T$,

$$\left[\lambda_j\right]^T\left[\delta\dot{\xi}_i\right] \doteq \left[\lambda_j\right]^T\left[\frac{\partial\dot{\xi}_i}{\partial\xi_j}\right]\left[\delta\xi_j\right] + \left[\lambda_j\right]^T\left[\frac{\partial\dot{\xi}_i}{\partial\nu_j}\right]\left[\delta\nu_j\right] \tag{5-4}$$

$$\left[\delta\xi_j\right]^T\left[\dot{\lambda}_i\right] = -\left[\delta\xi_j\right]^T\left[\frac{\partial\dot{\xi}_i}{\partial\xi_j}\right]^T\left[\lambda_j\right] \tag{5-5}$$

Adding equations (5-4) and (5-5) yields

$$\left[\delta\xi_j\right]^T\left[\dot{\lambda}_i\right] + \left[\lambda_j\right]^T\left[\delta\dot{\xi}_i\right] \doteq \left[\lambda_j\right]^T\left[\frac{\partial\dot{\xi}_i}{\partial\nu_j}\right]\left[\delta\nu_j\right] \tag{5-6}$$

Note that the matrix formed by the product

$$\left[\lambda_j\right]^T\left[\frac{\partial\dot{\xi}_i}{\partial\xi_j}\right]\left[\delta\xi_j\right]$$

is identically equal to the transpose of the matrix formed by

$$\left[\delta\xi_j\right]^T\left[\frac{\partial\dot{\xi}_i}{\partial\xi_j}\right]^T\left[\lambda_j\right]$$

However, both of these multiplications produce matrices containing only a single element (first row, first column). In this case the matrix and its transpose are equal and their difference is zero.

Equation (5-6) can now be rewritten in the expended form.

$$\left\{\begin{aligned}&\lambda_1\delta\dot{h} + \lambda_2\delta\dot{\theta} + \lambda_3\delta\dot{\gamma} + \lambda_4\delta\dot{V}\\ &+\dot{\lambda}_1\delta h + \dot{\lambda}_2\delta\theta + \dot{\lambda}_3\delta\gamma + \dot{\lambda}_4\delta V\end{aligned}\right\} \doteq \lambda_3\frac{\partial\dot{\gamma}}{\partial\alpha}\delta\alpha + \lambda_4\frac{\partial\dot{V}}{\partial T}\delta T \qquad (5\text{-}7)$$

For a constant thrust, gravity turn all elements of $\left[\frac{\partial\dot{\xi}_i}{\partial\nu_j}\right]$ except $\frac{\partial\dot{\gamma}}{\partial\alpha}$ and $\frac{\partial\dot{V}}{\partial T}$ are zero.

Regrouping the expressions on the left side of equation (5-7), and integrating with respect to time from $t = t_1$ to $t = t_f$ yields

$$\int_{t_1}^{t_f}\Big[(\lambda_1\delta\dot{h} + \dot{\lambda}_1\delta h) + (\lambda_2\delta\dot{\theta} + \dot{\lambda}_2\delta\theta) + (\lambda_3\delta\dot{\gamma} + \dot{\lambda}_3\delta\gamma) + (\lambda_4\delta\dot{V} + \dot{\lambda}_4\delta V)\Big]dt \doteq \int_{t_1}^{t_f}\left[\lambda_3\frac{\partial\dot{\gamma}}{\partial\alpha}\delta\alpha + \lambda_4\frac{\partial\dot{V}}{\partial T}\delta T\right]dt \qquad (5\text{-}8)$$

$$0 \leq t_1 \leq t_f$$

Since the operator, δ, and the differentiation indicated by the dot are commutative, each pair of products enclosed within parentheses forms a perfect differential such that the left hand integral can be evaluated as a function of end point values.

$$\left[\lambda_1\delta h + \lambda_2\delta\theta + \lambda_3\delta\gamma + \lambda_4\delta V\right]_{t_f} \doteq \left[\lambda_1\delta h + \lambda_2\delta\theta + \lambda_3\delta\gamma + \lambda_4\delta V\right]_{t_1} + \int_{t_1}^{t_f}\left[\lambda_3\frac{\partial\dot{\gamma}}{\partial\alpha}\delta\alpha + \lambda_4\frac{\partial\dot{V}}{\partial T}\delta T\right]dt \qquad (5\text{-}9)$$

To solve equation (5-9) for the final position deviations, which are δh and $\delta\theta$ evaluated at $t = t_f$, proceed as follows. Integrate the set of adjoint equations (5-3) backward in time from $t = t_f$ to determine the values of λ_1, λ_2, λ_3, and λ_4 at $t = t_1$. This backward integration is performed twice, using two different sets of values for $[\lambda_j(t_f)]$ as initial conditions. The first set is denoted $[{}_h\lambda_j\ (t_f)]$, and yields a solution for $\delta h\ (t_f)$. The second set, $[{}_\theta\lambda_j\ (t_f)]$, yields a solution for $\delta\theta\ (t_f)$. In particular

$$\left[{}_h\lambda_j(t_f)\right] = \begin{bmatrix} {}_h\lambda_1(t_f) \\ {}_h\lambda_2(t_f) \\ {}_h\lambda_3(t_f) \\ {}_h\lambda_4(t_f) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and (5-10)

$$\left[{}_\theta\lambda_j(t_f)\right] = \begin{bmatrix} {}_\theta\lambda_1(t_f) \\ {}_\theta\lambda_2(t_f) \\ {}_\theta\lambda_3(t_f) \\ {}_\theta\lambda_4(t_f) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying equations (5-10) to equation (5-9) results in:

$$(\delta h)_{t_f} \doteq \left({}_h\lambda_1\,\delta h + {}_h\lambda_2\,\delta\theta + {}_h\lambda_3\,\delta\gamma + {}_h\lambda_4\,\delta V\right)_{t_1} + \int_{t_1}^{t_f}\left({}_h\lambda_3\frac{\partial\dot{\gamma}}{\partial\alpha}\,\delta\alpha + {}_h\lambda_4\frac{\partial\dot{V}}{\partial T}\,\delta T\right)dt$$

and (5-11)

$$(\delta\theta)_{t_f} \doteq \left({}_\theta\lambda_1\,\delta h + {}_\theta\lambda_2\,\delta\theta + {}_\theta\lambda_3\,\delta\gamma + {}_\theta\lambda_4\,\delta V\right)_{t_1} + \int_{t_1}^{t_f}\left({}_\theta\lambda_3\frac{\partial\dot{\gamma}}{\partial\alpha}\,\delta\alpha + {}_\theta\lambda_4\frac{\partial\dot{V}}{\partial T}\,\delta T\right)dt$$

The symbols ${}_h\lambda_j$ and ${}_\Theta\lambda_j$ refer to values of λ_j computed from backward integrations using the final conditions ${}_h\lambda_j\ (t_f)$ and ${}_\Theta\lambda_j\ (t_f)$ respectively. Final deviations in the remaining two state variables, γ and V. can be found by performing backward integrations of equations (5-3) for two more sets of final conditions. Namely,

$$\left[{}_\gamma\lambda_j(t_f)\right] = \begin{bmatrix} {}_\gamma\lambda_1(t_f) \\ {}_\gamma\lambda_2(t_f) \\ {}_\gamma\lambda_3(t_f) \\ {}_\gamma\lambda_4(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad (5\text{-}12)$$

and

$$\left[{}_v\lambda_j(t_f)\right] = \begin{bmatrix} {}_v\lambda_1(t_f) \\ {}_v\lambda_2(t_f) \\ {}_v\lambda_3(t_f) \\ {}_v\lambda_4(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{(5-12 cont.)}$$

The control errors δT and $\delta\alpha$ are considered constants over the trajectory and can therefore be removed from the integrals. This step, along with the following substitutions allow equations (5-11) to be expressed in a more convenient form.

Let

$$E_{\delta h} \equiv \left({}_h\lambda_1\,\delta h + {}_h\lambda_2\,\delta\theta + {}_h\lambda_3\,\delta\gamma + {}_h\lambda_4\,\delta V\right)_{t_1}$$

$$E_{\delta\theta} \equiv \left({}_\theta\lambda_1\,\delta h + {}_\theta\lambda_2\,\delta\theta + {}_\theta\lambda_3\,\delta\gamma + {}_\theta\lambda_4\,\delta V\right)_{t_1}$$

$$I_{Th} \equiv \int_{t_1}^{t_f} {}_h\lambda_4 \frac{\partial \dot{V}}{\partial T}\,dt \qquad I_{\alpha h} \equiv \int_{t_1}^{t_f} {}_h\lambda_3 \frac{\partial \dot{\gamma}}{\partial \alpha}\,dt \qquad \text{(5-13)}$$

$$I_{T\theta} \equiv \int_{t_1}^{t_f} {}_\theta\lambda_4 \frac{\partial \dot{V}}{\partial T}\,dt \qquad I_{\alpha\theta} \equiv \int_{t_1}^{t_f} {}_\theta\lambda_3 \frac{\partial \dot{\gamma}}{\partial \alpha}\,dt$$

Then,

$$\delta h(t_f) \doteq E_{\delta h} + I_{Th}\,\delta T + I_{\alpha h}\,\delta\alpha$$

$$\delta\theta(t_f) \doteq E_{\delta\theta} + I_{T\theta}\,\delta T + I_{\alpha\theta}\,\delta\alpha \qquad \text{(5-14)}$$

It is desired to find the values of δT and $\delta\alpha$ that will reduce the final position errors to zero. These are found by solving the following set of simultaneous equations.

$$E_{\delta h} + I_{Th}\,\delta T + I_{\alpha h}\,\delta\alpha \doteq 0$$

$$E_{\delta\theta} + I_{T\theta}\,\delta T + I_{\alpha\theta}\,\delta\alpha \doteq 0 \qquad \text{(5-15)}$$

The solutions are:

$$\delta T \doteq \frac{E_{S\theta} I_{\alpha h} - E_{Sh} I_{\alpha\theta}}{I_{Th} I_{\alpha h} - I_{T\theta} I_{\alpha h}}$$

and (5-16)

$$\delta\alpha \doteq \frac{E_{Sh} I_{T\theta} - E_{S\theta} I_{Th}}{I_{Th} I_{\alpha\theta} - I_{T\theta} I_{\alpha h}}$$

Expanding the quantities $E_{S\theta}$ and E_{Sh} produces,

$$(\delta T)_{t_1} \doteq \left\{ \frac{\left[I_{\alpha h}\, {}_{\theta}\lambda_1 - I_{\alpha\theta}\, {}_{h}\lambda_1\right]_{t_1} \delta h(t_1) + \left[I_{\alpha h}\, {}_{\theta}\lambda_2 - I_{\alpha\theta}\, {}_{h}\lambda_2\right]_{t_1} \delta\theta(t_1)}{D(t_1)} \right.$$

$$\left. + \frac{\left[I_{\alpha h}\, {}_{\theta}\lambda_3 - I_{\alpha\theta}\, {}_{h}\lambda_3\right]_{t_1} \delta\gamma(t_1) + \left[I_{\alpha h}\, {}_{\theta}\lambda_4 - I_{\alpha\theta}\, {}_{h}\lambda_4\right]_{t_1} \delta V(t_1)}{D(t_1)} \right\}$$

and (5-17)

$$(\delta\alpha)_{t_1} \doteq \left\{ \frac{\left[I_{T\theta}\, {}_{h}\lambda_1 - I_{Th}\, {}_{\theta}\lambda_1\right]_{t_1} \delta h(t_1) + \left[I_{T\theta}\, {}_{h}\lambda_2 - I_{Th}\, {}_{\theta}\lambda_2\right]_{t_1} \delta\theta(t_1)}{D(t_1)} \right.$$

$$\left. + \frac{\left[I_{T\theta}\, {}_{h}\lambda_3 - I_{Th}\, {}_{\theta}\lambda_3\right]_{t_1} \delta\gamma(t_1) + \left[I_{T\theta}\, {}_{h}\lambda_4 - I_{Th}\, {}_{\theta}\lambda_4\right]_{t_1} \delta V(t_1)}{D(t_1)} \right\}$$

Where,

$$D(t_1) \equiv \left[I_{Th} I_{\alpha\theta} - I_{T\theta} I_{\alpha h}\right]_{t_1}$$

The guidance terms are defined from this pair of equations.

$$S_1(t_1) \equiv \frac{\left[I_{\alpha h}\, {}_{\theta}\lambda_1 - I_{\alpha\theta}\, {}_{h}\lambda_1\right]_{t_1}}{D(t_1)}$$

$$S_2(t_1) \equiv \frac{\left[I_{\alpha h}\, {}_{\theta}\lambda_2 - I_{\alpha\theta}\, {}_{h}\lambda_2\right]_{t_1}}{D(t_1)}$$

$$S_3(t_1) \equiv \frac{\left[I_{\alpha h}\, {}_{\theta}\lambda_3 - I_{\alpha\theta}\, {}_{h}\lambda_3\right]_{t_1}}{D(t_1)} \qquad (5\text{-}19)$$

$$S_4(t_1) \equiv \frac{\left[I_{\alpha h}\, {}_{\theta}\lambda_4 - I_{\alpha\theta}\, {}_{h}\lambda_4\right]_{t_1}}{D(t_1)}$$

$$S_5 \equiv \frac{\left[I_{T\theta}\,k\lambda_1 - I_{Th}\,\theta\lambda_1\right]_{t_1}}{D(t_1)}$$

$$S_6 \equiv \frac{\left[I_{T\theta}\,k\lambda_2 - I_{Th}\,\theta\lambda_2\right]_{t_1}}{D(t_1)} \qquad (5\text{-}19 \text{ cont.})$$

$$S_7 \equiv \frac{\left[I_{T\theta}\,k\lambda_3 - I_{Th}\,\theta\lambda_3\right]_{t_1}}{D(t_1)}$$

$$S_8 \equiv \frac{\left[I_{T\theta}\,k\lambda_4 - I_{Th}\,\theta\lambda_4\right]_{t_1}}{D(t_1)}$$

Equations (5-17) can then be expressed as

$$\delta T(t_1) = \left[S_1\,\delta h + S_2\,\delta\theta + S_3\,\delta\gamma + S_4\,\delta V\right]_{t_1}$$

$$\delta\alpha(t_1) = \left[S_5\,\delta h + S_6\,\delta\theta + S_7\,\delta\gamma + S_8\,\delta V\right]_{t_1}$$

The guidance terms are precomputed from the reference trajectory and stored in the guidance computer along the reference values of the state variables.

6.0 APPENDIX B: DERIVATION OF EQUATIONS RELATING STATE VARIABLES TO RADAR DATA

Figure 1 illustrates the geometrical situation existing for the descent phase. The symbols that will be used are defined as follows.

$\gamma \equiv$ The angle between the velocity vector and the local horizontal plane measured in the plane of motion, (flight path angle).

$h \equiv$ Altitude of vehicle above a reference lunar sphere of radius r_o.

$\Theta \equiv$ The angular displacement of the vehicle from the landing site in moon centered coordinates. (measured in the plane of motion.)

$V \equiv$ Magnitude of space vehicle velocity vector.

$R \equiv$ Line-of-sight range from vehicle to landing site (point A).

$\dot{R} \equiv$ Time derivative of R.

$\phi \equiv$ Angle between local vertical and the line-of-sight to the landing site (point A).

$\dot{\phi} \equiv$ Time derivative of ϕ.

r_o = Radius of moon at landing site.

The landing site at point A is allowed to be in the plane formed by the velocity vector and local vertical.

To derive expressions relating the state variables h, Θ, γ, and V to the radar observables R, $\dot{R}$, ϕ, and $\dot{\phi}$; proceed as follows. The known quantities are R, $\dot{R}$, ϕ, $\dot{\phi}$, and r_o. The angle Θ can be computed by applying the law of sines to triangle AOB.

$$\frac{\sin \Theta}{R} = \frac{\sin \phi}{r_o}$$

$$\therefore \quad \Theta = \sin^{-1}\left[\frac{R \sin \phi}{r_o}\right] \qquad (6-1)$$

Then the quantity h is available from the law of sines.

$$\frac{r_0 + h}{\sin[\pi - (\phi + \theta)]} = \frac{r_0}{\sin\phi}$$

$$r_0 + h = \frac{r_0 \sin(\phi + \theta)}{\sin\phi}$$

$$h = \frac{r_0[\sin\phi\cos\theta + \cos\phi\sin\theta]}{\sin\phi} - r_0$$

$$h = r_0\cos\theta + \frac{r_0\sin\theta}{\tan\phi} - r_0$$

$$\therefore h = R\cos\phi + [r_0^2 - R^2\sin^2\phi]^{1/2} - r_0 \qquad (6\text{-}2)$$

Functions relating γ and V to the observables are most conveniently derived through vector analysis. In the following analysis an arrow over a quantity refers to a vector. and the quantity is the vector magnitude. For example, $\vec{V}$ is the velocity vector and $V = |\vec{V}|$.

The following definitions refer to figure 15.

$\vec{A} \equiv$ A vector from point O to point A.

$\vec{B} \equiv$ A vector from point O to point B.

$\vec{R} \equiv$ A vector from point B to point A.

$\vec{V} \equiv$ The velocity of point B.

In vector notation,

$$\vec{R} = \vec{A} - \vec{B} \qquad (6\text{-}3)$$

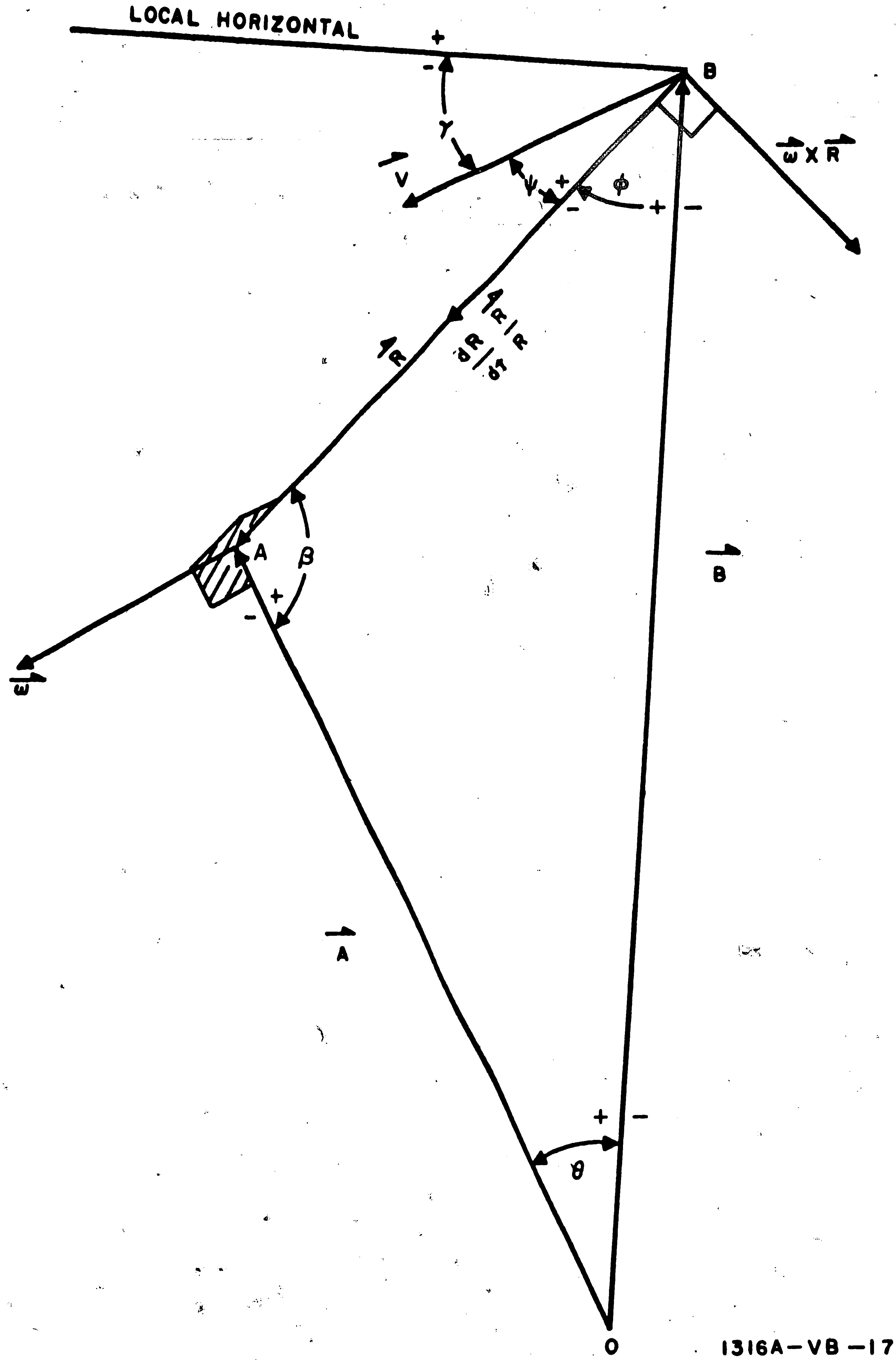

Figure 15. Vector Diagram of Landing Maneuver.

and

$$\dot{\vec{R}} = \dot{\vec{A}} - \dot{\vec{B}} \qquad (6-4)$$

Since $\vec{A}$ is a constant vector, $\dot{\vec{A}}$ is zero, and

$$\dot{\vec{R}} = -\dot{\vec{B}} = -\dot{\vec{V}} \qquad (6-5)$$

Vector differentiation results in the expression

$$\dot{\vec{R}} = \frac{dR}{dt}\frac{\vec{R}}{R} + \vec{\omega} \times \vec{R} \qquad (6-6)$$

$\frac{\vec{R}}{R}$ is a unit vector in the direction of R

The vector $\vec{\omega}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{R}$ and equal in magnitude to $\dot{\beta}$. For the geometry of figure 15, $\vec{\omega}$ is positive toward the reader. The quantity $\vec{\omega} \times \vec{R}$ is perpendicular to $\vec{R}$ and in the plane formed by $\vec{A}$ and $\vec{R}$. The positive senses of all vector quantities are illustrated in figure 15. The following statements can be made.

$$\frac{dR}{dt} = \dot{R} = \text{an observable}$$

$$\beta = \pi - (\phi + \theta)$$

$$\omega = \dot{\beta} = -\dot{\phi} - \dot{\theta}$$

where $\dot{\phi}$ is an observable and $\dot{\theta}$ as a function of observables can be obtained by differentiation of the expression obtained earlier for θ.

$$\theta = \sin^{-1}\left[\frac{R}{r_0}\sin\phi\right]$$

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{R\dot{\phi}\cos\phi + \dot{R}\sin\phi}{\left[r_0^2 - R^2\sin^2\phi\right]^{1/2}} \qquad (6-7)$$

Therefore, if

$$\dot{\vec{R}} = \dot{R}\frac{\vec{R}}{R} + \vec{\omega} \times \vec{R}$$

then

$$\vec{V} = -\dot{\vec{R}} = -\dot{R}\frac{\vec{R}}{R} - \vec{\omega} \times \vec{R}$$

$$V^2 = (\dot{R})^2 + R^2(\dot{\phi} + \dot{\theta})^2$$

$$V = \left[(\dot{R})^2 + R^2(\dot{\phi} + \dot{\theta})^2\right]^{1/2} \tag{6-8}$$

and

$$\tan\psi = \tan\left[-\tan^{-1}\left(\frac{\omega R}{\dot{R}}\right)\right] = -\frac{\omega R}{\dot{R}}$$

$$\tan\psi = \frac{R(\dot{\phi} + \dot{\theta})}{\dot{R}}$$

From figure 15 it can be shown that

$$\phi + \psi - \gamma = \pi/2$$

$$\gamma = \phi + \psi - \pi/2$$

$$\gamma = \phi - \frac{\pi}{2} + \tan^{-1}\left[\frac{R(\dot{\phi} + \dot{\theta})}{\dot{R}}\right] \tag{6-9}$$

The four resulting transformation equations are:

$$\theta = \sin^{-1}\left[\frac{R}{r_0}\sin\phi\right] \tag{6-10}$$

$$h = R\cos\phi + \left[r_0^2 - R^2\sin^2\phi\right]^{1/2} - r_0$$

$$\gamma = \phi - \frac{\pi}{2} + \tan^{-1}\left[\frac{R(\dot{\phi} + \dot{\theta})}{\dot{R}}\right]$$

(6-10 cont.)

$$V = \left[(\dot{R})^2 + R^2(\dot{\phi} + \dot{\theta})^2\right]^{1/2}$$

where

$$\dot{\theta} = \frac{R\dot{\phi}\cos\phi + \dot{R}\sin\phi}{\left[r_0^2 - R^2\sin^2\phi\right]^{1/2}}$$

The term $\dot{\theta}$ appearing in the expressions for γ and V appreciably complicates the partial derivative equations required for the linearized error analysis. However, over the range of parameter values covered by the landing maneuver, $\dot{\theta}$ is very much smaller than $\dot{\phi}$ so that the approximation $\dot{\phi} + \dot{\theta} \doteq \dot{\phi}$ is very accurate. The equations from which the partial derivatives were derived are then:

$$\theta = \sin^{-1}\left[\frac{R}{r_0}\sin\phi\right]$$

$$h = R\cos\phi + \left[r_0^2 - R^2\sin^2\phi\right]^{1/2} - r_0$$

$$\gamma \doteq \phi - \pi/2 + \tan^{-1}\left[\frac{R\dot{\phi}}{\dot{R}}\right] \qquad (6\text{-}11)$$

$$V \doteq \left[(\dot{R})^2 + (R\dot{\phi})^2\right]^{1/2}$$

The resulting partial derivative expressions which are used in the analysis are:

$$\frac{\partial h}{\partial R} = \cos\phi - \frac{R\sin^2\phi}{\left[r_0^2 - R^2\sin^2\phi\right]^{1/2}} \qquad (6\text{-}12)$$

$$\frac{\partial \theta}{\partial R} = \frac{\sin \phi}{\left[r_0^2 - R^2 \sin^2 \phi\right]^{1/2}}$$

$$\frac{\partial \gamma}{\partial R} \doteq \frac{\dot{R}\dot{\phi}}{\left[(\dot{R})^2 + (R\dot{\phi})^2\right]}$$

$$\frac{\partial V}{\partial R} \doteq \frac{R\dot{\phi}^2}{\left[(\dot{R})^2 + (R\dot{\phi})^2\right]^{1/2}}$$

$$\frac{\partial h}{\partial \dot{R}} = 0$$

$$\frac{\partial \theta}{\partial \dot{R}} = 0 \qquad \text{(6-12 cont.)}$$

$$\frac{\partial \gamma}{\partial \dot{R}} \doteq \frac{-1}{(\dot{R})^2 + (R\dot{\phi})^2}$$

$$\frac{\partial V}{\partial \dot{R}} \doteq \frac{\dot{R}}{\left[(\dot{R})^2 + (R\dot{\phi})^2\right]^{1/2}}$$

$$\frac{\partial h}{\partial \phi} = -R \sin \phi \left\{1 + \frac{R\cos\phi}{\left[r_0^2 - R^2 \sin^2 \phi\right]^{1/2}}\right\}$$

$$\frac{\partial \theta}{\partial \phi} = \frac{R\cos\phi}{\left[r_0^2 - R^2 \sin^2 \phi\right]^{1/2}}$$

$$\frac{\partial \gamma}{\partial \phi} \doteq 1$$

$$\frac{\partial V}{\partial \phi} \doteq 0$$

$$\frac{\partial h}{\partial \dot{\phi}} = 0$$

$$\frac{\partial \theta}{\partial \dot{\phi}} = 0$$

$$\frac{\partial \gamma}{\partial \dot{\phi}} \doteq \frac{R\dot{R}}{\left[(\dot{R})^2 + (R\dot{\phi})^2\right]}$$

$$\frac{\partial V}{\partial \dot{\phi}} \doteq \frac{R^2\dot{\phi}}{\left[(\dot{R})^2 + (R\dot{\phi})^2\right]^{1/2}}$$

7.0 APPENDIX C: PARTIAL DERIVATIVES RELATING THRUST VECTOR ERRORS TO ERRORS IN $\ddot{h}$, $\ddot{\theta}$, AND $\dot{V}$

To determine the required partial derivatives it is first necessary to obtain equations for $\ddot{h}$, $\ddot{\theta}$, and $\dot{V}$ in terms of the thrust quantities T and α. The quantity $\ddot{h}$ is the component of acceleration in the positive h direction. With the aid of figure 3, $\ddot{h}$ is seen to be

$$\ddot{h} = -\frac{T}{m}\sin(\gamma+\alpha) - \frac{\mu}{(r_0+h)^2} \tag{7-1}$$

In a similar manner expressions for $\ddot{\theta}$ and $\dot{V}$ are available.

$$\ddot{\theta} = \frac{T\cos(\gamma+\alpha)}{m(r_0+h)} \tag{7-2}$$

$$\dot{V} = -\frac{T\cos\alpha}{m} - \frac{\mu\sin\gamma}{(r_0+h)^2} \tag{7-3}$$

Partial differentiations of these equations with respect to T and α yields the desired quantities.

$$\frac{\partial\dot{V}}{\partial T} = -\frac{\cos\alpha}{m} \qquad \frac{\partial\dot{V}}{\partial\alpha} = \frac{T\sin\alpha}{m}$$

$$\frac{\partial\ddot{h}}{\partial T} = -\frac{\sin(\gamma+\alpha)}{m} \qquad \frac{\partial\ddot{h}}{\partial\alpha} = -\frac{T\cos(\gamma+\alpha)}{m} \tag{7-4}$$

$$\frac{\partial\ddot{\theta}}{\partial T} = \frac{\cos(\gamma+\alpha)}{m(r_0+h)} \qquad \frac{\partial\ddot{\theta}}{\partial\alpha} = -\frac{T\sin(\gamma+\alpha)}{m(r_0+h)}$$

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